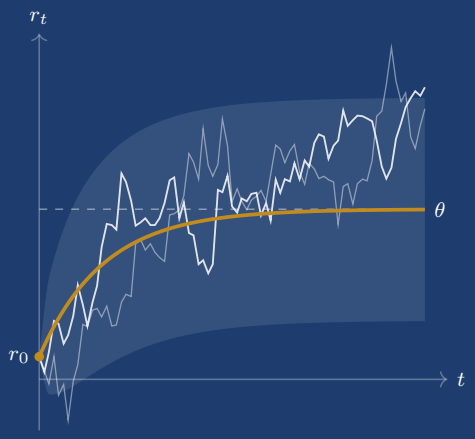
UNIVERSITÉ PARIS 1
PANTHÉON SORBONNE

Interest Rate Models

From short rates to market models

Lecture Notes & Exercises



Master MMMEF & IRFA
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How to use these notes. These notes follow the course *Financial Products and Introduction to Pricing*, whose lecture notes are referred to as [FP]; the results recalled from [FP] are flagged as such. Notations follow [2] as much as possible. Exercises are gathered at the end of each section and their solutions at the end of the document.

Unless otherwise specified, time is measured in years and rates are expressed per annum. Zero-coupon rates are continuously compounded, money-market and forward (LIBOR-type) rates are simply compounded. A swap or a swaption is called *payer* (resp. *receiver*) when its holder pays (resp. receives) the fixed rate. The main notations are summarised in Table 1.

Symbol	Meaning
$r(t)$	instantaneous short rate
$B(t) = \exp\left(\int_0^t r(s)\,ds\right)$	bank (money-market) account
$D(t,T) = B(t)/B(T)$	stochastic discount factor between t and T
$P(t,T)$	price at t of the zero-coupon bond paying 1 at T, $P(T,T) = 1$
$\tau(t,T)$	year fraction between t and T (day-count convention)
$R(t,T)$	continuously compounded zero-coupon rate, $P(t,T) = e^{-R(t,T)\tau(t,T)}$
$L(t,T)$	simply compounded (money-market) rate, $P(t,T) = (1+\tau(t,T)L(t,T))^{-1}$
$F(t;T,S)$	simply compounded forward rate between T and S, seen from t
$f(t,T)$	instantaneous forward rate
$\mathcal{T} = \{T_\alpha, \dots, T_\beta\},\ \tau_i$	schedule of dates, $\tau_i = \tau(T_{i-1}, T_i)$
$F_k(t) = F(t;T_{k-1},T_k)$	k-th forward rate of the schedule
$C_{\alpha,\beta}(t) = \sum_{i=\alpha+1}^{\beta} \tau_i P(t,T_i)$	annuity of the schedule
$S_{\alpha,\beta}(t)$	forward swap rate
$\mathbb{Q},\ \mathbb{Q}^T,\ \mathbb{Q}^{\alpha,\beta}$	risk-neutral, T-forward and swap measures
$\mathbb{E},\ \mathbb{E}^T,\ \mathbb{E}^{\alpha,\beta}$	corresponding expectations
$\mathcal{F}_t$	information available at date t (filtration)
$N(\cdot),\ n(\cdot)$	cumulative distribution and density of the standard normal law
$\mathrm{Bl}(F,K,v)$	Black's formula, $FN(d_1) - KN(d_2)$, $d_{1,2} = \frac{\ln(F/K) \pm v^2/2}{v}$

Table 1: Main notations

1 Products and notations

This section introduces the interest rate products studied in the course and fixes the notations. The first two subsections recall material from [FP]; swaps, caps and swaptions are new.

1.1 Zero-coupon bonds and interest rates (recall from [FP])

A **zero-coupon bond** of maturity T pays one unit of currency at date T and nothing before. Its price at date $t \le T$ is denoted $P(t,T)$, with $P(T,T) = 1$. A deterministic stream of cash flows c_i paid at dates T_i is a portfolio of zero-coupon bonds and is worth $\sum_i c_i P(t,T_i)$: the **discount curve** $T \mapsto P(t,T)$ summarises all the information on interest rates available at date t.

Interest rates are only conventions to express $P(t,T)$. They involve a **year fraction** $\tau(t,T)$ computed with a day-count convention: act/360 for EUR and USD money-market rates, act/365 for GBP and for continuously compounded rates, 30/360 for the fixed leg of EUR swaps. For instance, from January 1 to July 1, 2027 (181 days), $\tau = 181/360 = 0.5028$ in act/360 and $\tau = 181/365 = 0.4959$ in act/365.

Definition 1. *The **continuously compounded** and the **simply compounded** zero-coupon rates are defined by*

$$R(t,T) := -\frac{\ln P(t,T)}{\tau(t,T)}, \qquad L(t,T) := \frac{1}{\tau(t,T)}\left(\frac{1}{P(t,T)} - 1\right),$$

*that is $P(t,T) = e^{-R(t,T)\tau(t,T)} = (1+\tau(t,T)L(t,T))^{-1}$. The **zero-coupon curve** (or yield curve) at date t is the graph of*

$$T \mapsto \begin{cases} L(t,T) & t < T \le t+1 \quad \textit{(money market)}, \\ R(t,T) & T > t+1. \end{cases}$$

Money-market rates (deposits, Euribor, LIBOR) are simply compounded. For maturities beyond one year, the market quotes annually compounded rates $Y(t,T)$, defined by $P(t,T) = (1+Y(t,T))^{-\tau(t,T)}$; we use continuously compounded rates, which are easier to manipulate. The same discount factor gives different numbers: for $P = 0.97$ and $\tau = 1$, $L = Y = 3.093\%$ while $R = 3.046\%$.

1.2 Forward rate agreements and forward rates (recall from [FP])

Definition 2. *A **Forward Rate Agreement** (FRA) involves three dates: the current date t, the fixing date $T > t$ and the payment date $S > T$. For a notional N and a fixed rate K, the holder receives at S the fixed amount $N\tau(T,S)K$ and pays the floating amount $N\tau(T,S)L(T,S)$, where the rate $L(T,S)$ is fixed at T. The payoff at S is therefore*

$$N\tau(T,S)\left(K - L(T,S)\right).$$

An FRA locks in, as of today, the rate at which one can lend between T and S. The cash flows are displayed in Figure 1. In this subsection, $\tau = \tau(T,S)$.

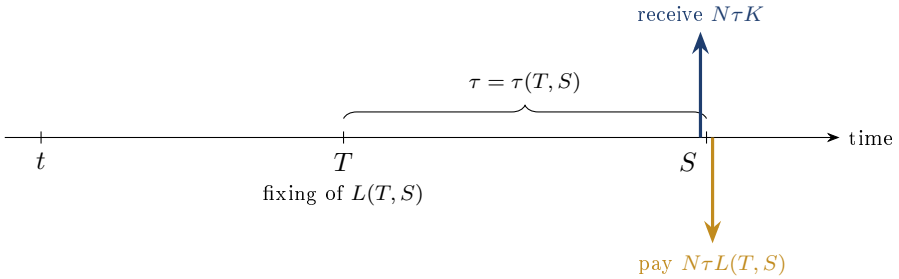


Figure 1: Cash flows of an FRA (receiving the fixed rate)

Valuation. The fixed leg is a known amount paid at S: it is worth $N\tau K P(t,S)$. The floating leg can be replicated without any model. Since $\tau L(T,S) = 1/P(T,S) - 1$, receiving $N\tau L(T,S)$ at S is the same as receiving $N/P(T,S) - N$ at S, which is obtained as follows:

- at t, buy N zero-coupon bonds of maturity T and sell N zero-coupon bonds of maturity S, for a cost $N\left(P(t,T) - P(t,S)\right)$;

- at T, lend the N received until S at the rate $L(T,S)$;
- at S, receive $N\,(1+\tau L(T,S)) = N/P(T,S)$ and pay N.

The floating leg is therefore worth $N\,(P(t,T)-P(t,S))$. In particular, a floating leg to which the notional is added at S is worth N at the fixing date T: *a floating rate note quotes at par at its reset dates.*

Definition 3. *The* ***forward rate*** $F(t;T,S)$ *is the fixed rate which makes the FRA worth zero at date* t*:*

$$\tau F(t;T,S)P(t,S) = P(t,T) - P(t,S), \tag{1}$$

$$F(t;T,S) = \frac{1}{\tau}\left(\frac{P(t,T)}{P(t,S)} - 1\right). \tag{2}$$

The value of the FRA is then $N\tau P(t,S)\,(K - F(t;T,S))$, and $F(T;T,S) = L(T,S)$: the forward rate converges to the rate which is eventually fixed.

Definition 4. *The* ***instantaneous forward rate*** *is*

$$f(t,T) \coloneqq \lim_{S\to T^+} F(t;T,S) = -\frac{\partial \ln P(t,T)}{\partial T}, \quad \textit{so that} \quad P(t,T) = \exp\left(-\int_t^T f(t,u)\,du\right).$$

With $\tau(t,T) = T-t$, the zero-coupon rate $R(t,T) = \frac{1}{T-t}\int_t^T f(t,u)\,du$ is the average of the instantaneous forward rates: forward rates are above zero-coupon rates when the curve is upward sloping.

1.3 Interest rate swaps

Let $\mathcal{T} = \{T_\alpha, T_{\alpha+1}, \dots, T_\beta\}$ be a schedule of dates and $\tau_i = \tau(T_{i-1}, T_i)$. An **interest rate swap** (IRS) exchanges a fixed leg against a floating leg on this schedule: at each date T_i, $i = \alpha+1, \dots, \beta$,

- the fixed leg pays $N\tau_i K$;
- the floating leg pays $N\tau_i L(T_{i-1}, T_i)$, the rate being fixed at T_{i-1} and paid at T_i.

The first fixing date T_α can be today (spot-starting swap) or in the future (forward-starting swap). In a **payer** swap (PFS) one pays the fixed leg and receives the floating leg; in a **receiver** swap (RFS) one receives the fixed leg (Figure 2). In practice the two legs have different frequencies and day counts; we use the same schedule for simplicity.

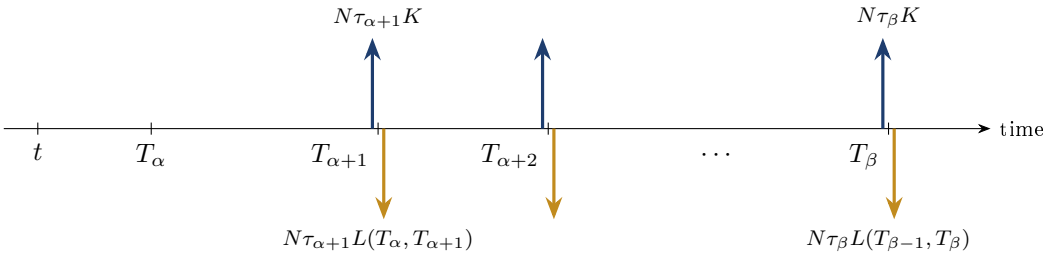


Figure 2: Cash flows of a receiver swap: fixed leg received (up), floating leg paid (down)

Valuation. A receiver swap is a portfolio of FRAs. Writing $F_i(t) = F(t;T_{i-1},T_i)$ and using (1), which gives $\sum_{i=\alpha+1}^{\beta}\tau_i P(t,T_i)F_i(t) = \sum_{i=\alpha+1}^{\beta}(P(t,T_{i-1}) - P(t,T_i)) = P(t,T_\alpha) - P(t,T_\beta)$,

$$\begin{aligned} RFS(t) &= N\sum_{i=\alpha+1}^{\beta}\tau_i P(t,T_i)\,(K - F_i(t)) \\ &= N\left[K\,C_{\alpha,\beta}(t) - (P(t,T_\alpha) - P(t,T_\beta))\right], \qquad C_{\alpha,\beta}(t) \coloneqq \sum_{i=\alpha+1}^{\beta}\tau_i P(t,T_i). \end{aligned} \tag{3}$$

The **annuity** $C_{\alpha,\beta}(t)$ is the value of a fixed leg paying one unit of rate per year. The fixed leg is a coupon-bearing bond without its final notional, and the floating leg is worth $N\,(P(t,T_\alpha) - P(t,T_\beta))$: it is a floating rate note without its final notional.

Definition 5. *The **forward swap rate** $S_{\alpha,\beta}(t)$ is the fixed rate which makes the swap worth zero at date t:*

$$S_{\alpha,\beta}(t) = \frac{P(t,T_\alpha) - P(t,T_\beta)}{C_{\alpha,\beta}(t)}. \tag{4}$$

Then $RFS(t) = NC_{\alpha,\beta}(t)\,(K - S_{\alpha,\beta}(t))$ and $PFS(t) = NC_{\alpha,\beta}(t)\,(S_{\alpha,\beta}(t) - K)$.

Proposition 1. *The forward swap rate is a weighted average of the forward rates:*

$$S_{\alpha,\beta}(t) = \sum_{i=\alpha+1}^{\beta} w_i(t) F_i(t), \qquad w_i(t) = \frac{\tau_i P(t,T_i)}{C_{\alpha,\beta}(t)}, \tag{5}$$

with positive weights summing to one. In terms of forward rates only,

$$S_{\alpha,\beta}(t) = \frac{1 - \prod_{j=\alpha+1}^{\beta} \frac{1}{1+\tau_j F_j(t)}}{\sum_{i=\alpha+1}^{\beta} \tau_i \prod_{j=\alpha+1}^{i} \frac{1}{1+\tau_j F_j(t)}}.$$

Proof. The first formula is the telescoping identity used above, divided by $C_{\alpha,\beta}(t)$. For the second one, divide the numerator and the denominator of (4) by $P(t,T_\alpha)$ and use $\frac{P(t,T_k)}{P(t,T_\alpha)} = \prod_{j=\alpha+1}^{k} \frac{P(t,T_j)}{P(t,T_{j-1})} = \prod_{j=\alpha+1}^{k} \frac{1}{1+\tau_j F_j(t)}$. □

The weights w_i depend only mildly on the level of rates. Freezing them is the basis of a useful approximation for swaptions in Section 5.

1.4 Caps and floors

Definition 6. *A **caplet** on $[T_{i-1}, T_i]$ with strike K pays at T_i*

$$N\tau_i\,(L(T_{i-1},T_i) - K)^+.$$

*A **cap** is a sum of caplets on the dates of a schedule; its discounted payoff is*

$$\sum_{i=\alpha+1}^{\beta} D(t,T_i) N\tau_i\,(L(T_{i-1},T_i) - K)^+,$$

where $D(t,T_i)$ is the stochastic discount factor of Section 2. Floorlets and floors are defined in the same way with the payoff $N\tau_i\,(K - L(T_{i-1},T_i))^+$.

A borrower who pays a floating rate and buys a cap never pays more than K. Since $x^+ - (-x)^+ = x$, a cap minus a floor with the same strike is a payer swap (**cap-floor parity**). Each caplet depends on a single rate, which makes caps relatively easy to price. A caplet is also an option on a zero-coupon bond:

Proposition 2. *The value at the fixing date T_{i-1} of the caplet payoff is that of $N(1+\tau_i K)$ puts on the zero-coupon bond $P(T_{i-1},T_i)$ with strike $1/(1+\tau_i K)$:*

$$N\tau_i P(T_{i-1},T_i)\,(L(T_{i-1},T_i) - K)^+ = N(1+\tau_i K)\left(\frac{1}{1+\tau_i K} - P(T_{i-1},T_i)\right)^+. \tag{6}$$

The proof is left as Exercise 1.3. A model which prices options on zero-coupon bonds therefore prices caps (Section 4).

1.5 Swaptions

Definition 7. *A European **payer swaption** with maturity T_α and strike K gives the right, not the obligation, to enter at T_α the payer swap of schedule $\mathcal{T}$ and fixed rate K. The length $T_\beta - T_\alpha$ of the underlying swap is the **tenor**; a "$1y \times 5y$" swaption is an option of maturity one year on a five-year swap.*

The swaption is exercised when the swap has a positive value. Its payoff at T_α is

$$N\left(\sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha,T_i)\,(F_i(T_\alpha) - K)\right)^+ = NC_{\alpha,\beta}(T_\alpha)\,(S_{\alpha,\beta}(T_\alpha) - K)^+. \tag{7}$$

Symmetrically, a receiver swaption pays $NC_{\alpha,\beta}(T_\alpha)\,(K - S_{\alpha,\beta}(T_\alpha))^+$. Since $KC_{\alpha,\beta}(T_\alpha) - (1 - P(T_\alpha,T_\beta)) = \sum_i c_i P(T_\alpha,T_i) - 1$ with $c_i = \tau_i K$ for $i < \beta$ and $c_\beta = 1 + \tau_\beta K$, a receiver swaption is a put with strike 1 on the coupon bond paying c_i at T_i:

$$\text{receiver swaption payoff} = N\left(\sum_{i=\alpha+1}^{\beta} c_i P(T_\alpha,T_i) - 1\right)^+. \tag{8}$$

Caps versus swaptions. The sum in (7) is inside the positive part. Since $x \mapsto x^+$ is convex and positively homogeneous, $\left(\sum_i x_i\right)^+ \leq \sum_i x_i^+$, hence

$$\left(\sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha, T_i)\left(F_i(T_\alpha) - K\right)\right)^+ \leq \sum_{i=\alpha+1}^{\beta} \tau_i P(T_\alpha, T_i)\left(F_i(T_\alpha) - K\right)^+,$$

with no equality in general. The right-hand side is itself smaller than the value at T_α of the caplets of the cap (Exercise 1.3), so that a payer swaption is cheaper than the corresponding cap. More importantly, the swaption payoff cannot be split into payoffs depending on one rate each: its value depends on the *joint* behaviour of the forward rates $F_{\alpha+1}, \ldots, F_\beta$, and in particular on their correlations. This is the main difference between the two classes of products. In the market, caps are quoted through the Black volatility of forward rates (Section 5) and swaptions through the Black volatility of the swap rate (Section 3).

Exercises

Exercise 1.1 *Day count and rates*

A deposit of EUR 1,000,000 is made from January 15 to July 15, 2027 (181 days) at the money-market rate of 2.5% (act/360).

(a) Compute the interest paid at maturity.

(b) Compute the equivalent simply compounded rate in act/365.

(c) Compute the discount factor and the continuously compounded rate (act/365).

Exercise 1.2 *Swap rate and forward rates*

The discount factors are $P(0,1) = 0.97$, $P(0,2) = 0.94$ and $P(0,3) = 0.91$. Consider a three-year swap with annual payments ($\tau_i = 1$).

(a) Compute the forward rates $F_1(0), F_2(0), F_3(0)$.

(b) Compute the annuity and the three-year swap rate.

(c) Check Proposition 1 numerically.

(d) Compute the value of a receiver swap of notional 100 and fixed rate 3%.

(e) Show that the swap rate always lies between the smallest and the largest forward rate. When is it equal to the forward rates?

Exercise 1.3 *Caplets as options on zero-coupon bonds, parities*

(a) Prove Proposition 2.

(b) Show that a cap minus a floor with the same strike and schedule is a payer swap.

(c) Show that a payer swaption minus a receiver swaption with the same strike is a forward-starting payer swap, worth $NC_{\alpha,\beta}(t)\left(S_{\alpha,\beta}(t) - K\right)$.

(d) Using Jensen's inequality under the forward measures of Section 3, show that the value at T_α of the caplet paying at T_i is larger than $N\tau_i P(T_\alpha, T_i)\left(F_i(T_\alpha) - K\right)^+$. Deduce that a payer swaption is cheaper than the cap with the same strike and schedule.

Exercise 1.4 *Instantaneous forward rates*

Assume that $R(0,T) = a + \frac{b}{2}T$ with $a = 2\%$ and $b = 0.4\%$ and $\tau(0,T) = T$.

(a) Compute $f(0,T)$ and compare it with $R(0,T)$.

(b) Compute the forward rate $F(0;1,2)$ and compare it with $f(0,1.5)$.

2 Pricing with stochastic interest rates

When interest rates are deterministic, discounting a payoff paid at T simply means multiplying its expected value by $P(0,T)$. When rates are random, the discount factor itself becomes random. This section shows how the pricing formula of [FP] extends to this situation.

2.1 One payment date (recall from [FP])

Consider claims paying a single random amount at date T, and assume that they belong to $\mathcal{X} = L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ (finite variance). Two securities are traded: the zero-coupon bond of maturity T, worth $P(0,T)$, and a stock paying no dividend, worth S_0 today and S_T at T.

Cash and carry. Buying the stock at date 0 with money borrowed until T produces $S_T - S_0/P(0,T)$ at T. Selling at the same time a forward contract at the price F_T produces the riskless amount $F_T - S_0/P(0,T)$ for a zero initial cost. Absence of arbitrage therefore imposes $F_T = S_0/P(0,T)$.

Pricing functional. Let $\pi(X)$ be the price at date 0 of a claim $X \in \mathcal{X}$. Absence of arbitrage implies that π is linear and that a non-negative non-zero payoff has a positive price; assume moreover that π is continuous. By the Riesz representation theorem, there exists $\Psi \in \mathcal{X}$, the *state price density*, such that $\pi(X) = \mathbb{E}(\Psi X)$, and positivity of π makes Ψ positive. Setting $k_T = \mathbb{E}(\Psi)$ and $d\mathbb{Q}^T/d\mathbb{P} = \Psi/\mathbb{E}(\Psi)$ defines a probability measure $\mathbb{Q}^T$ such that

$$\pi(X) = k_T \, \mathbb{E}^T(X). \tag{9}$$

Applying (9) to $X = 1$ gives $k_T = P(0,T)$, and applying it to the forward contract, whose price is zero, gives

$$\mathbb{E}^T(S_T) = \frac{S_0}{P(0,T)} = F_T.$$

The probability $\mathbb{Q}^T$ depends on the payment date T: it is the *T-forward measure* studied in Section 3.

2.2 Several dates: the stochastic discount factor

Formula (9) involves one measure per payment date. To obtain a single pricing measure, consider a series of dates $t_0 < t_1 < \cdots < t_n$, typically days, and a one-period rate known at the beginning of each period. The **rolled-over discount factor**

$$D(t_0, t_k) := \prod_{j=0}^{k-1} P(t_j, t_{j+1})$$

is such that $1/D(t_0,t_k)$ is the value at t_k of one unit invested at t_0 and rolled over from period to period.[1] Seen from t_0, $D(t_0,t_k)$ is random since the future one-period rates are unknown, but $D(t_0,t_{k+1})$ is known at date t_k.

On each period, the one-date argument applies conditionally on the information at t_k. We now assume that the same probability works for all periods:

(H) There exists a probability $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that, for every k,

$$\mathbb{E}^{\mathbb{Q}}\left[S(t_{k+1}) \mid \mathcal{F}_{t_k}\right] = \frac{S(t_k)}{P(t_k, t_{k+1})}.$$

Rolling cash and carry. At each date t_k, enter a one-period cash and carry on $D(t_0,t_{k+1})$ shares: buy them, financed by borrowing $D(t_0,t_{k+1})S(t_k)$ until t_{k+1}. The amount to repay at t_{k+1} is

$$\frac{D(t_0,t_{k+1})S(t_k)}{P(t_k,t_{k+1})} = D(t_0,t_k)S(t_k),$$

so that the strategy generates the following flows:

Date	Flow
t_1	$D(t_0,t_1)S(t_1) - S(t_0)$
$\vdots$	$\vdots$
t_{k+1}	$D(t_0,t_{k+1})S(t_{k+1}) - D(t_0,t_k)S(t_k)$
$\vdots$	$\vdots$
t_n	$D(t_0,t_n)S(t_n) - D(t_0,t_{n-1})S(t_{n-1})$

[1] It is denoted $B(t_0,t_k)$ in [FP]. Here B is reserved for the bank account.

The crucial point is that the number of shares $D(t_0, t_{k+1})$ is known at t_k, when the trade is entered.

Proposition 3. *Under (H),*

$$S(t_0) = \mathbb{E}^{\mathbb{Q}}\left[D(t_0, t_n)S(t_n) \mid \mathcal{F}_{t_0}\right]. \tag{10}$$

Proof. Multiplying (H) by $D(t_0, t_{k+1})$, which is $\mathcal{F}_{t_k}$-measurable,

$$\mathbb{E}^{\mathbb{Q}}\left[D(t_0, t_{k+1})S(t_{k+1}) - D(t_0, t_k)S(t_k) \mid \mathcal{F}_{t_k}\right] = 0.$$

Taking the conditional expectation with respect to $\mathcal{F}_{t_0}$ (tower property) and summing over $k = 0, \ldots, n-1$, the intermediate terms cancel and $\mathbb{E}^{\mathbb{Q}}\left[D(t_0, t_n)S(t_n) - S(t_0) \mid \mathcal{F}_{t_0}\right] = 0$. □

Formulas (9) and (10) both express prices as expected discounted values, with a crucial difference. In (9) the discount factor $P(0,T)$ is known at the inception of the strategy and stands outside the expectation. In (10) the financing cost of the successive cash and carry strategies is not known at inception: the discount factor is stochastic and stands inside the expectation.

2.3 Continuous time: the bank account numeraire

Write $P(t_k, t_{k+1}) = \exp\left(-R(t_k, t_{k+1})(t_{k+1} - t_k)\right)$. When the time step goes to zero, the one-period rate $R(t_k, t_{k+1})$ tends to the **instantaneous short rate** $r(t_k)$ and

$$D(t_0, t_n) = \exp\left(-\sum_{k=0}^{n-1} R(t_k, t_{k+1})(t_{k+1} - t_k)\right) \longrightarrow D(t,T) = \exp\left(-\int_t^T r(s)\,ds\right) = \frac{B(t)}{B(T)},$$

where $B(t) = \exp\left(\int_0^t r(s)\,ds\right)$ is the **bank account**: $dB(t) = r(t)B(t)\,dt$. The bank account has no Brownian part: it is locally riskless, but its value over a finite period is random.

Proposition 4 (Pricing with stochastic interest rates)**.** *In a market without arbitrage opportunities, there exists a probability $\mathbb{Q}$ equivalent to $\mathbb{P}$, the **risk-neutral measure**, such that, for any traded asset paying no dividend, the two following equivalent properties hold:*

1. *the price is the expectation of the discounted value:*

$$S_t = \mathbb{E}^{\mathbb{Q}}\left[D(t,T)S_T \mid \mathcal{F}_t\right]; \tag{11}$$

2. *prices expressed in units of bank account are martingales: $\frac{S_t}{B(t)} = \mathbb{E}^{\mathbb{Q}}\left[\frac{S_T}{B(T)} \mid \mathcal{F}_t\right]$.*

In particular, a payoff X_T paid at T is worth $X_t = \mathbb{E}^{\mathbb{Q}}\left[D(t,T)X_T \mid \mathcal{F}_t\right]$ at $t \le T$.

The two properties are equivalent because $B(t)$ is known at t and $D(t,T) = B(t)/B(T)$. This statement is the version of the fundamental theorem of asset pricing used in these notes: absence of arbitrage (in a suitable technical sense) is equivalent to the existence of such a measure. Heuristically, if S/B had a predictable drift, a strategy trading in the direction of this drift would be an arbitrage.

2.4 Consequences

Zero-coupon bonds. Applying (11) to the zero-coupon bond, whose final value is $P(T,T) = 1$,

$$P(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)\,ds\right) \mid \mathcal{F}_t\right]. \tag{12}$$

The whole zero-coupon curve is therefore determined by the dynamics of the short rate under $\mathbb{Q}$. This is the starting point of short-rate models (Section 4).

Drift of traded assets. Let S be the price of a traded asset paying no dividend, with $dS_t = S_t\left(\mu_t\,dt + \sigma_t\,dW_t\right)$ where W is a $\mathbb{Q}$-Brownian motion. Since B has no Brownian part,

$$d\left(\frac{S_t}{B(t)}\right) = \frac{1}{B(t)}dS_t - \frac{S_t}{B(t)^2}dB(t) = \frac{S_t}{B(t)}\left(\left(\mu_t - r(t)\right)dt + \sigma_t\,dW_t\right).$$

A continuous martingale has no drift, hence $\mu_t = r(t)$: *under $\mathbb{Q}$, all traded assets paying no dividend have a drift equal to the short rate.*

Deterministic rates. When r is deterministic, $D(t,T) = P(t,T)$ and (11) reduces to (9): the risk-neutral and forward measures coincide. With stochastic rates they differ, and switching between them is the object of Section 3.

Exercises

Exercise 2.1 *Deterministic interest rates*

Assume that the short rate $r(t)$ is a deterministic function of time.

(a) Show that $P(t,T) = \exp\left(-\int_t^T r(s)\,ds\right)$ and that $f(t,T) = r(T)$ for all $t \le T$.

(b) Show that the forward rates are realised: $F(t;T,S) = L(T,S)$ for $t \le T$.

Exercise 2.2 *Cost of carry*

Under $\mathbb{Q}$, the price of an underlying follows $dS_t = S_t\,(c\,dt + \sigma\,dW_t)$ where c is the cost of carry, and the interest rate r is constant.

(a) Discuss the value of c with respect to r when S is a stock paying a continuous dividend yield q, an exchange rate, and a commodity.

(b) Show that the forward price of maturity T is $F_T = S_0 e^{cT}$.

Exercise 2.3 *Forward rate and expected future rate*

Consider two annual periods. The one-year rate is $r_0 = 2\%$ today; in one year it will be $r_1 = 1\%$ or $r_1 = 3\%$ with probability $1/2$ each under $\mathbb{Q}$. Rates are simply compounded, so that $P(t_k, t_{k+1}) = 1/(1 + r_k)$.

(a) Compute $P(0,1)$ and $P(0,2)$ using Proposition 3. Compare $P(0,2)$ with $1/1.02^2$.

(b) Compute the forward rate $F(0;1,2)$ and compare it with $\mathbb{E}^{\mathbb{Q}}(r_1)$.

(c) Define $\mathbb{Q}^2$ by its density $D(0,2)/P(0,2)$ with respect to $\mathbb{Q}$. Show that $\mathbb{E}^{\mathbb{Q}^2}(r_1) = F(0;1,2)$.

3 Change of measure and change of numeraire

The risk-neutral measure of Section 2 is one pricing measure among many. Each choice of a reference asset, the *numeraire*, comes with its own pricing measure, and choosing the right one often makes a pricing problem easy. This section provides the tools: conditional expectations under a change of measure, Girsanov's theorem, and the change of numeraire formula.

3.1 Conditional expectation as a projection in L^2

The space $H = L^2(\Omega, \mathcal{F}, \mathbb{P})$ of square integrable random variables is a Hilbert space for the scalar product $\langle X, Y\rangle = \mathbb{E}(XY)$. Let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra (the information available at some date) and $G = L^2(\Omega, \mathcal{G}, \mathbb{P})$, a closed subspace of H. The orthogonal projection $p(X)$ of X on G is the unique element of G such that

$$\forall Y \in G, \quad \langle X - p(X), Y\rangle = 0 \quad \Longleftrightarrow \quad \forall Y \in G, \quad \langle X, Y\rangle = \langle p(X), Y\rangle\,.$$

In probabilistic terms, $p(X) = \mathbb{E}(X \mid \mathcal{G})$: the conditional expectation is the best prediction of X in the mean square sense given the information $\mathcal{G}$ (Figure 3).

Now let $\mathbb{Q}$ be a probability equivalent to $\mathbb{P}$ with density $Z = d\mathbb{Q}/d\mathbb{P}$ (positive, with $\mathbb{E}(Z) = 1$); for simplicity, assume that all the random variables below are bounded. The $\mathbb{Q}$-scalar product is

$$\langle X, Y\rangle_{\mathbb{Q}} = \mathbb{E}^{\mathbb{Q}}(XY) = \mathbb{E}(ZXY) = \langle ZX, Y\rangle\,,$$

and the projection $q(X) = \mathbb{E}^{\mathbb{Q}}(X \mid \mathcal{G})$ for this scalar product satisfies, for all $Y \in G$,

$$\langle X - q(X), Y\rangle_{\mathbb{Q}} = 0 \Longleftrightarrow \langle ZX, Y\rangle = \langle Z, q(X)Y\rangle \Longleftrightarrow \langle p(ZX), Y\rangle = \langle p(Z)q(X), Y\rangle\,,$$

where the last step uses $q(X)Y \in G$. Since $p(Z)q(X)$ belongs to G and the projection is unique, $p(ZX) = p(Z)q(X)$, that is $q(X) = p(ZX)/p(Z)$.

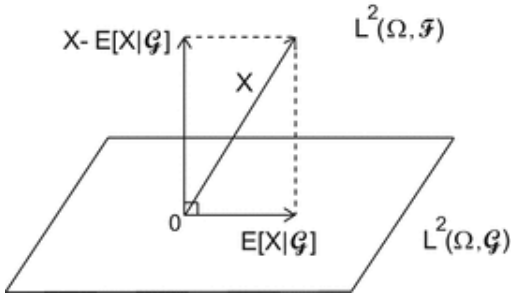


Figure 3: Conditional expectation as a projection

Theorem 1 (Bayes rule)**.** *Let $\mathbb{Q}$ be a probability equivalent to $\mathbb{P}$ with density $Z = d\mathbb{Q}/d\mathbb{P}$, X a $\mathbb{Q}$-integrable random variable and $\mathcal{G}$ a sub-σ-algebra. Then:*

1. *the density of $\mathbb{Q}$ with respect to $\mathbb{P}$ restricted to $\mathcal{G}$ is $\frac{d\mathbb{Q}}{d\mathbb{P}}\big|_{\mathcal{G}} = \mathbb{E}(Z \mid \mathcal{G})$;*
2. $\mathbb{E}^{\mathbb{Q}}(X \mid \mathcal{G}) = \dfrac{\mathbb{E}(ZX \mid \mathcal{G})}{\mathbb{E}(Z \mid \mathcal{G})}$.

3.2 Girsanov's theorem

Let $(\mathcal{F}_t)_{t\in[0,T]}$ be a filtration and $\mathbb{Q}$ a probability equivalent to $\mathbb{P}$ on $\mathcal{F}_T$, with density Z_T. The **density process**

$$Z_t = \mathbb{E}(Z_T \mid \mathcal{F}_t) = \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t}, \qquad Z_0 = 1,$$

is a positive $\mathbb{P}$-martingale. By the Bayes rule, $X_t = \mathbb{E}^{\mathbb{Q}}(X_T \mid \mathcal{F}_t)$ if and only if $X_t Z_t = \mathbb{E}(Z_T X_T \mid \mathcal{F}_t)$, hence:

Proposition 5. *A process X is a $\mathbb{Q}$-martingale if and only if XZ is a $\mathbb{P}$-martingale.*

When the filtration is generated by a Brownian motion W, the martingale representation theorem shows that the positive martingale Z can be written $dZ_t = Z_t\lambda_t\,dW_t$ for some adapted process λ, that is

$$Z_t = \mathcal{E}(\lambda \cdot W)_t := \exp\left(\int_0^t \lambda_s\,dW_s - \frac{1}{2}\int_0^t \lambda_s^2\,ds\right),$$

the **Doléans-Dade exponential** of $\int \lambda\,dW$. When λ is deterministic, $\int_0^t \lambda_s\,dW_s$ is Gaussian with mean zero and variance $\int_0^t \lambda_s^2 ds$. Conversely, starting from a process λ, $\mathcal{E}(\lambda \cdot W)$ is a positive local martingale; it is a true martingale, and defines a probability, under the Novikov condition $\mathbb{E}\exp\left(\frac{1}{2}\int_0^T \lambda_s^2 ds\right) < \infty$, which is an *assumption* on λ.

Theorem 2 (Girsanov)**.** *Let M be a continuous $\mathbb{P}$-martingale and Z the (continuous) density process of $\mathbb{Q}$. Then*

$$N_t = M_t - \int_0^t \frac{d\langle M, Z\rangle_s}{Z_s} = M_t - \langle M, \ln Z\rangle_t$$

is a $\mathbb{Q}$-(local) martingale.

Proof. Write $N = M + \Lambda$ with Λ of finite variation, so that $\langle \Lambda, Z\rangle = 0$. By Itô's formula,

$$d(NZ) = N\,dZ + Z\,dM + Z\,d\Lambda + d\langle M, Z\rangle.$$

The first two terms are martingale increments. By Proposition 5, N is a $\mathbb{Q}$-martingale if and only if the finite variation part vanishes: $Z\,d\Lambda + d\langle M, Z\rangle = 0$, that is $d\Lambda = -d\langle M, Z\rangle / Z$. Finally, $d\ln Z = dZ/Z - \frac{1}{2}d\langle Z\rangle / Z^2$ gives $d\langle M, \ln Z\rangle = d\langle M, Z\rangle / Z$. □

Corollary 1. *Assume $dZ_t = Z_t\lambda_t\,dW_t$, and let W' be a $\mathbb{P}$-Brownian motion with $d\langle W, W'\rangle_t = \rho_t\,dt$. Then*

$$W'_t - \int_0^t \rho_s\lambda_s\,ds$$

is a $\mathbb{Q}$-Brownian motion. In particular, $W_t - \int_0^t \lambda_s\,ds$ is a $\mathbb{Q}$-Brownian motion.

Proof. $d\langle W', \ln Z\rangle_t = \lambda_t\,d\langle W', W\rangle_t = \rho_t\lambda_t\,dt$. By Theorem 2, the process is a continuous $\mathbb{Q}$-local martingale with quadratic variation t (quadratic variations do not depend on the equivalent measure), hence a $\mathbb{Q}$-Brownian motion by Lévy's characterisation. □

In practice: *the drift of a process under $\mathbb{Q}$ is its drift under $\mathbb{P}$ plus $d\langle X, \ln Z\rangle_t / dt$.*

3.3 Change of numeraire

A **numeraire** is a traded asset paying no dividend with a positive price, used as a unit of account. In Proposition 4, the numeraire is the bank account B and the associated measure is $\mathbb{Q}$.

Proposition 6 (Change of numeraire)**.** *Let N^1 and N^2 be two numeraires and $\mathbb{Q}^1$ a probability such that X/N^1 is a $\mathbb{Q}^1$-martingale for every traded asset X. Define $\mathbb{Q}^2$ by the density process*

$$Z_t = \left.\frac{d\mathbb{Q}^2}{d\mathbb{Q}^1}\right|_{\mathcal{F}_t} = \frac{N_0^1}{N_0^2}\frac{N_t^2}{N_t^1}. \tag{13}$$

Then X/N^2 is a $\mathbb{Q}^2$-martingale for every traded asset X. Consequently, a payoff X_T paid at T is worth, for any numeraire N with associated measure $\mathbb{Q}^N$,

$$X_t = N_t\,\mathbb{E}^N\left[\frac{X_T}{N_T} \mid \mathcal{F}_t\right]. \tag{14}$$

Proof. N^2/N^1 is a traded asset expressed in units of N^1, hence a positive $\mathbb{Q}^1$-martingale, and $Z_0 = 1$: Z is a density process. For a traded asset X, $\frac{X}{N^2}Z = \frac{N_0^1}{N_0^2}\frac{X}{N^1}$ is a $\mathbb{Q}^1$-martingale, so X/N^2 is a $\mathbb{Q}^2$-martingale by Proposition 5. □

The guiding principle is to choose the numeraire which makes the payoff simple. Three numeraires are used in these notes: the zero-coupon bond (for payoffs paid at a single date), the annuity (for swaptions) and the foreign bank account (for quanto products).

3.4 The T-forward measure

Take the zero-coupon bond $P(\cdot,T)$ as numeraire. By (13) with $N^1 = B$, the T-**forward measure** $\mathbb{Q}^T$ has density

$$\left.\frac{d\mathbb{Q}^T}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = \frac{D(0,t)P(t,T)}{P(0,T)}, \qquad \left.\frac{d\mathbb{Q}^T}{d\mathbb{Q}}\right|_{\mathcal{F}_T} = \frac{D(0,T)}{P(0,T)}. \tag{15}$$

The density on $\mathcal{F}_T$ is positive and has expectation one since $P(0,T) = \mathbb{E}^{\mathbb{Q}}[D(0,T)]$. It is the measure $\mathbb{Q}^T$ met in Section 2.

Proposition 7. *Let $\mathbb{E}^T$ denote the expectation under $\mathbb{Q}^T$.*

1. *A payoff X_T paid at T is worth $X_t = P(t,T)\,\mathbb{E}^T[X_T \mid \mathcal{F}_t]$ at $t \le T$.*
2. *Forward prices $X_t/P(t,T)$ of traded assets are $\mathbb{Q}^T$-martingales. In particular, the forward rate $F(t;T,S)$ is a $\mathbb{Q}^S$-martingale and*
$$\mathbb{E}^S[L(T,S) \mid \mathcal{F}_t] = F(t;T,S). \tag{16}$$
3. *The instantaneous forward rate is the expectation of the future short rate under the forward measure:*
$$f(t,T) = \mathbb{E}^T[r(T) \mid \mathcal{F}_t]. \tag{17}$$

Proof. 1 and the first part of 2 follow from (14) with $N = P(\cdot,T)$, since $P(T,T) = 1$. By (1), $1+\tau F(t;T,S) = P(t,T)/P(t,S)$ is the forward price of the zero-coupon bond of maturity T for delivery at S, hence a $\mathbb{Q}^S$-martingale, and $F(T;T,S) = L(T,S)$. For 3, differentiate (12) with respect to T and use point 1:

$$\partial_T P(t,T) = -\mathbb{E}^{\mathbb{Q}}[D(t,T)r(T) \mid \mathcal{F}_t] = -P(t,T)\,\mathbb{E}^T[r(T) \mid \mathcal{F}_t],$$

and $f(t,T) = -\partial_T P(t,T)/P(t,T)$. □

Under $\mathbb{Q}$, one models $X_t/B(t)$, whose final value $X_T/B(T)$ mixes the payoff and the path of rates. Under $\mathbb{Q}^T$, the discounting is done outside the expectation and only the distribution of X_T matters. Note also that forward rates are *not* risk-neutral expectations of future rates, but expectations under the appropriate forward measure (Exercise 2.3).

Proposition 8 (Black's formula)**.** *Let X be a traded asset whose forward price $F_t = X_t/P(t,T)$ satisfies $dF_t = F_t\sigma(t)\,dW_t^T$ under $\mathbb{Q}^T$, with σ deterministic. Then the call paying $(X_T - K)^+$ at T is worth*

$$P(t,T)\ \mathrm{Bl}(F_t,K,v) = P(t,T)\,(F_tN(d_1) - KN(d_2)), \qquad d_{1,2} = \frac{\ln(F_t/K) \pm v^2/2}{v}, \quad v^2 = \int_t^T \sigma(s)^2 ds, \tag{18}$$

and the put is worth $P(t,T)\,(KN(-d_2) - F_tN(-d_1))$.

Proof. $X_T = F_T$ and, conditionally on $\mathcal{F}_t$, $\ln F_T$ is Gaussian with mean $\ln F_t - v^2/2$ and variance v^2. The computation of $\mathbb{E}^T\left[(F_T - K)^+ \mid \mathcal{F}_t\right]$ is that of the Black-Scholes formula with zero interest rate (Exercise 3.1). □

This is the route to the prices of caplets (Section 5) and of options on zero-coupon bonds (Section 4).

3.5 The swap measure

The annuity $C_{\alpha,\beta}(t) = \sum_{i=\alpha+1}^{\beta} \tau_i P(t,T_i)$ is a positive portfolio of zero-coupon bonds, hence a numeraire. Its measure $\mathbb{Q}^{\alpha,\beta}$ is the **swap measure**. By (4), the forward swap rate is the price of the traded portfolio $P(\cdot,T_\alpha) - P(\cdot,T_\beta)$ in units of the annuity: *$S_{\alpha,\beta}$ is a $\mathbb{Q}^{\alpha,\beta}$-martingale.*

By (14) and (7), the payer swaption is worth

$$PS(t) = C_{\alpha,\beta}(t)\,\mathbb{E}^{\alpha,\beta}\left[\frac{NC_{\alpha,\beta}(T_\alpha)\,(S_{\alpha,\beta}(T_\alpha) - K)^+}{C_{\alpha,\beta}(T_\alpha)} \mid \mathcal{F}_t\right] = NC_{\alpha,\beta}(t)\,\mathbb{E}^{\alpha,\beta}\left[(S_{\alpha,\beta}(T_\alpha) - K)^+ \mid \mathcal{F}_t\right].$$

The annuity cancels: the swaption is a call on the swap rate, which is a martingale under the swap measure. In the **lognormal forward-swap model** (LSM), $dS_{\alpha,\beta}(t) = \sigma_{\alpha,\beta} S_{\alpha,\beta}(t)\,dW_t^{\alpha,\beta}$ under $\mathbb{Q}^{\alpha,\beta}$, which gives **Black's formula for swaptions**:

$$PS(t) = NC_{\alpha,\beta}(t)\ \mathrm{Bl}\left(S_{\alpha,\beta}(t), K, \sigma_{\alpha,\beta}\sqrt{T_\alpha - t}\right). \tag{19}$$

Swaptions are quoted in the market through the volatility $\sigma_{\alpha,\beta}$ of this formula (or through a normal volatility, see Section 5). The LSM is not compatible with the lognormal model for forward rates used for caps (Section 5): if the F_i are lognormal, $S_{\alpha,\beta}$ is not exactly lognormal.

3.6 The foreign measure and the quanto adjustment

Consider a domestic currency (say EUR) and a foreign currency (say USD). Let X_t be the price in domestic currency of one unit of foreign currency (EUR per USD), B and B^f the domestic and foreign bank accounts, $\mathbb{Q}$ and $\mathbb{Q}^f$ the domestic and foreign risk-neutral measures, and S^f the price in foreign currency of a foreign asset paying no dividend. Interest rates are constant in this subsection.

The foreign measure. S^f/B^f is a $\mathbb{Q}^f$-martingale. Since foreign currency can be converted at no cost, XS^f is the price of a domestic traded asset and XS^f/B is a $\mathbb{Q}$-martingale. In particular, XB^f is a domestic numeraire, whose measure is $\mathbb{Q}^f$. By (13),

$$Z_t = \left.\frac{d\mathbb{Q}^f}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = \frac{X_t B^f(t)}{X_0 B(t)}. \tag{20}$$

Dynamics of the exchange rate. Let $dX_t = X_t\left(\mu_X\,dt + \sigma_X\,dW_t^X\right)$ with W^X a $\mathbb{Q}$-Brownian motion. The drift of XB^f/B is $\mu_X + r^f - r$, which must vanish: under $\mathbb{Q}$, $\mu_X = r - r^f$ (interest rate parity, see [FP]). Then $dZ_t = Z_t\sigma_X\,dW_t^X$.

Dynamics of foreign assets. Under $\mathbb{Q}^f$, $dS_t^f = S_t^f\left(r^f dt + \sigma_S\,dW_t^f\right)$ where W^f is a $\mathbb{Q}^f$-Brownian motion. Let W^S be the $\mathbb{Q}$-Brownian motion driving S^f under $\mathbb{Q}$, with $d\left\langle W^S, W^X\right\rangle_t = \rho\,dt$. By Corollary 1, $W_t^S - \rho\sigma_X t$ is a $\mathbb{Q}^f$-Brownian motion, which is W^f. Hence

$$dS_t^f = S_t^f\left(\left(r^f - \rho\sigma_S\sigma_X\right)dt + \sigma_S\,dW_t^S\right) \quad \text{under } \mathbb{Q}. \tag{21}$$

The same result follows from the absence of drift of XS^f/B under $\mathbb{Q}$: the drift of XS^f is $(r - r^f) + \mu_S + \rho\sigma_S\sigma_X$, which must equal r.

Quanto forward. A quanto contract pays at T the *number* S_T^f in domestic currency, without conversion. By (21), it is worth

$$e^{-rT}\mathbb{E}^{\mathbb{Q}}\left(S_T^f\right) = S_0^f\,e^{(r^f - r - \rho\sigma_S\sigma_X)T}. \tag{22}$$

The term $-\rho\sigma_S\sigma_X$ is the **quanto adjustment**. When $\rho > 0$, the foreign asset tends to rise when the foreign currency appreciates: its domestic value XS^f gains on both legs, and since this domestic value must grow at the rate r under $\mathbb{Q}$, the growth of S^f itself must be lower. The quanto payoff, which does not benefit from the currency gain, is therefore cheaper than the plain forward. For instance, receiving in one year as many EUR as the performance N_1/N_0 of the (USD) Nasdaq, with $r = r^f = 0$, $\sigma_S = 20\%$ and $\sigma_X = 10\%$, is worth $e^{-0.02\rho}$: from 1.0202 for $\rho = -1$ to 0.9802 for $\rho = 1$.

Exercises

Exercise 3.1 *Black-Scholes formula by change of numeraire*

Under $\mathbb{Q}$, $dS_t = S_t\,(c\,dt + \sigma\,dW_t)$ with S_0 given, and the interest rate r is constant. Let $F_T = \mathbb{E}^{\mathbb{Q}}(S_T)$ and $Y_t = \exp\left(-\frac{1}{2}\sigma^2 t + \sigma W_t\right)$. We want the price $C = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[(S_T - K)^+\right] = e^{-rT}(B - A)$ with $B = \mathbb{E}^{\mathbb{Q}}\left[S_T \mathbf{1}_{S_T \geq K}\right]$ and $A = K\mathbb{Q}(S_T \geq K)$.

(a) Integrate S, compute F_T and show that $S_T = F_T Y_T$.

(b) Show that $A = KN(d_A)$ with $d_A = \frac{\ln(F_T/K) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$.

(c) Show that $d\mathbb{Q}^S/d\mathbb{Q} = Y_T$ defines a probability equivalent to $\mathbb{Q}$, and that $W_t^S = W_t - \sigma t$ is a $\mathbb{Q}^S$-Brownian motion. Which numeraire does $\mathbb{Q}^S$ correspond to when $c = r$?

(d) Show that $B = F_T\,\mathbb{Q}^S(S_T \geq K) = F_T N(d_B)$ with $d_B = d_A + \sigma\sqrt{T}$.

(e) Deduce $C = e^{-rT}\,\mathrm{Bl}(F_T, K, \sigma\sqrt{T})$ and recover the Black-Scholes formula when $c = r$.

Exercise 3.2 *Exchange option (Margrabe formula)*

Under $\mathbb{Q}$, $dX_t = X_t\left(r\,dt + \sigma_X\,dW_t^X\right)$ and $dY_t = Y_t\left(r\,dt + \sigma_Y\,dW_t^Y\right)$ with $d\left\langle W^X, W^Y\right\rangle_t = \rho\,dt$, r constant. We want the price $V = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[(X_T - Y_T)^+\right]$ of the option to exchange Y for X.

(a) Show that

$$d\left(\frac{1}{Y_t}\right) = \frac{1}{Y_t}\left(-(r - \sigma_Y^2)\,dt - \sigma_Y\,dW_t^Y\right),$$
$$d\left(\frac{X_t}{Y_t}\right) = \frac{X_t}{Y_t}\left((\sigma_Y^2 - \rho\sigma_X\sigma_Y)\,dt + \sigma_X\,dW_t^X - \sigma_Y\,dW_t^Y\right).$$

(b) Let $\mathbb{Q}^Y$ be the measure of the numeraire Y. Give its density Z_T with respect to $\mathbb{Q}$ and show that $dZ_t = Z_t\sigma_Y\,dW_t^Y$.

(c) Show that $W_t^X - \rho\sigma_Y t$ and $W_t^Y - \sigma_Y t$ are $\mathbb{Q}^Y$-Brownian motions, and that X/Y is a $\mathbb{Q}^Y$-martingale with volatility $\sigma_{X/Y} = \sqrt{\sigma_X^2 + \sigma_Y^2 - 2\rho\sigma_X\sigma_Y}$.

(d) Show that $V = Y_0\,\mathbb{E}^Y\left[\left(\frac{X_T}{Y_T} - 1\right)^+\right] = X_0 N(d_1) - Y_0 N(d_2)$ with $d_{1,2} = \frac{\ln(X_0/Y_0) \pm \frac{1}{2}\sigma_{X/Y}^2 T}{\sigma_{X/Y}\sqrt{T}}$.

(e) Let X be the Nasdaq and Y the S&P 500, both with a 15% volatility, $X_0 = Y_0 = 1$ and $T = 1$. Give the price as a function of ρ and comment.

Exercise 3.3 *Quanto adjustment*

Use the notations of the last subsection.

(a) Explain why S^f/B^f is a $\mathbb{Q}^f$-martingale and XS^f/B a $\mathbb{Q}$-martingale. Show (20).

(b) Show that $\mu_X = r - r^f$ under $\mathbb{Q}$ and that $d\left(Z_t^{-1}\right) = Z_t^{-1}\left(\sigma_X^2\,dt - \sigma_X\,dW_t^X\right)$.

(c) Using Girsanov's theorem from $\mathbb{Q}^f$ to $\mathbb{Q}$ with the density Z^{-1}, show that $W_t^f + \rho\sigma_X t$ is a $\mathbb{Q}$-Brownian motion and recover (21).

(d) Let N_t be the USD value of the Nasdaq and $I_t = N_t/N_0$. What is the value today of receiving I_1 EUR in one year? Compute it as a function of ρ for $r = r^f = 0$, $\sigma_N = 20\%$ and $\sigma_X = 10\%$.

Exercise 3.4 *From one forward measure to another*

Let $T < S$ and $\tau = \tau(T, S)$.

(a) Show that $\left.\frac{d\mathbb{Q}^T}{d\mathbb{Q}^S}\right|_{\mathcal{F}_T} = \frac{1 + \tau L(T,S)}{1 + \tau F(0;T,S)}$.

(b) Deduce that $\mathbb{E}^T\left[L(T,S)\right] = F(0;T,S) + \frac{\tau\,\mathrm{Var}^S(L(T,S))}{1 + \tau F(0;T,S)}$. Is $F(\cdot;T,S)$ a $\mathbb{Q}^T$-martingale?

4 Short-rate models

By (12), the whole zero-coupon curve is determined by the dynamics of the short rate under $\mathbb{Q}$. Short-rate models specify this dynamics, $dr(t) = \mu(t, r(t))\,dt + \sigma(t, r(t))\,dW_t$ under $\mathbb{Q}$, and deduce bond prices from it. Historically, the purpose was not only to price derivatives but also to give a normative answer to the question of the possible shapes of the yield curve. Many models were proposed from the late 1970s onwards; Table 3.1 of [2] gives an overview. We start with the model of [6], the simplest and most influential one, and compute its bond prices in three different ways (Gaussian computation, PDE, forward measure), since each route generalises to other models. We then price options, fit the initial curve (Hull-White, Ho-Lee), keep rates positive (Cox-Ingersoll-Ross) and introduce several factors.

4.1 The Vasicek model

Under $\mathbb{Q}$, the short rate follows an Ornstein-Uhlenbeck process:

$$dr(t) = k\,(\theta - r(t))\,dt + \sigma\,dW_t, \qquad r(0) = r_0, \tag{23}$$

where W is a $\mathbb{Q}$-Brownian motion and k, θ, σ are positive constants:

- θ is the *long-term mean*: the trajectories of r fluctuate around θ in the long run;
- k is the *speed of mean reversion*: a deviation from θ is halved, on average, in $\ln 2/k$ years;
- σ is the *instantaneous volatility* of the short rate.

Vasicek specified the dynamics under the historical probability; with a constant market price of risk, the dynamics under $\mathbb{Q}$ is still of the form (23), with a modified long-term mean.

Solution. With $X(t) = r(t) - \theta$, $dX = -kX\,dt + \sigma\,dW$, and $Y(t) = e^{kt}X(t)$ satisfies $dY = \sigma e^{kt} dW$. Integrating between t and T,

$$r(T) = \theta + (r(t) - \theta)\,e^{-k(T-t)} + \sigma I(t,T), \qquad I(t,T) \coloneqq \int_t^T e^{-k(T-s)} dW_s. \tag{24}$$

The random variable $I(t,T)$ contains all the randomness of $r(T)$ seen from t, and is independent of $\mathcal{F}_t$.

Proposition 9. *Conditionally on $\mathcal{F}_t$, $(r(s))_{s \geq t}$ is a Gaussian process with*

$$\mathbb{E}\left[r(T) \mid \mathcal{F}_t\right] = \theta + (r(t) - \theta)\,e^{-k(T-t)}, \tag{25}$$

$$\mathrm{Var}\left(r(T) \mid \mathcal{F}_t\right) = \frac{\sigma^2}{2k}\left(1 - e^{-2k(T-t)}\right), \tag{26}$$

$$\gamma(t,s,u) \coloneqq \mathrm{Cov}\left(r(s), r(u) \mid \mathcal{F}_t\right) = \sigma^2 \mathbb{E}\left[I(t,s)I(t,u)\right] = \frac{\sigma^2}{2k} e^{-k(s+u)}\left(e^{2k(s\wedge u)} - e^{2kt}\right), \tag{27}$$

where $s \wedge u = \min(s,u)$.

Proof. $I(t,s)$ is a stochastic integral of a deterministic function: it is Gaussian with mean zero. By Itô's isometry, $\mathbb{E}\left[I(t,s)I(t,u)\right] = \int_t^{s\wedge u} e^{-k(s-v)} e^{-k(u-v)} dv = e^{-k(s+u)} \frac{e^{2k(s\wedge u)} - e^{2kt}}{2k}$, and $s = u = T$ gives the variance. □

Comments.

- When $k(T-t) \ll 1$, $\mathrm{Var}\left(r(T) \mid \mathcal{F}_t\right) \approx \sigma^2(T-t)$: on short horizons, the short rate behaves like a Brownian motion (the case $k = 0$).
- When $k(T-t) \gg 1$, the variance tends to $\sigma^2/(2k)$: mean reversion caps the uncertainty, and $r(T)$ converges in law to $\mathcal{N}\left(\theta, \sigma^2/(2k)\right)$.
- The initial value $r(t)$ is forgotten at an exponential speed.
- The short rate can be negative, with probability $N(-m/s)$ where m and s^2 are the conditional mean and variance. With $k = 0.3$, $\theta = 3\%$, $\sigma = 1\%$ and $r_0 = 1\%$, this probability is 4.0% in one year and 1.0% in the long run. Long seen as a flaw of the model, negative rates were observed on the euro between 2014 and 2022.

4.2 Route 1: the integrated short rate

By (12), $P(t,T) = \mathbb{E}\left[e^{-\xi(t,T)} \mid \mathcal{F}_t\right]$ with $\xi(t,T) = \int_t^T r(s)\,ds$. As an integral of a Gaussian process, $\xi(t,T)$ is Gaussian conditionally on $\mathcal{F}_t$, and the bond price follows from its mean and variance. Integrating (24),

$$\bar{\xi}(t,T) \coloneqq \mathbb{E}\left[\xi(t,T) \mid \mathcal{F}_t\right] = \theta(T-t) + (r(t)-\theta)\,A(t,T), \qquad A(t,T) \coloneqq \frac{1-e^{-k(T-t)}}{k}. \tag{28}$$

The variance is the double integral of the covariance:

$$V(t,T) \coloneqq \mathrm{Var}\left(\xi(t,T) \mid \mathcal{F}_t\right) = \int_t^T \int_t^T \gamma(t,s,u)\,ds\,du.$$

The integrand is symmetric in (s,u), and on the half-square $\{s<u\}$, $s \wedge u = s$. Hence

$$\begin{aligned} V(t,T) &= \frac{\sigma^2}{k}\int_t^T \left[\int_t^u e^{-k(s+u)}\left(e^{2ks}-e^{2kt}\right)ds\right]du = \frac{\sigma^2}{k}\int_t^T e^{-ku}\left[\int_t^u \left(e^{ks}-e^{2kt}e^{-ks}\right)ds\right]du \\ &= \frac{\sigma^2}{k^2}\int_t^T e^{-ku}\left[\left(e^{ku}-e^{kt}\right)+e^{2kt}\left(e^{-ku}-e^{-kt}\right)\right]du = \frac{\sigma^2}{k^2}\int_t^T \left[1-2e^{-k(u-t)}+e^{-2k(u-t)}\right]du. \end{aligned}$$

The three terms integrate to $T-t$, $-2A(t,T)$ and $\left(1-e^{-2k(T-t)}\right)/(2k)$, so that

$$V(t,T) = \frac{\sigma^2}{2k^3}\left[2k(T-t) - 3 + 4e^{-k(T-t)} - e^{-2k(T-t)}\right]. \tag{29}$$

Proposition 10. *In the Vasicek model,*

$$P(t,T) = e^{-\bar{\xi}(t,T)+\frac{1}{2}V(t,T)} = \exp\left(-A(t,T)r(t) - C(t,T)\right), \tag{30}$$

$$C(t,T) = \left(\theta - \frac{\sigma^2}{2k^2}\right)\left(T-t-A(t,T)\right) + \frac{\sigma^2}{4k}A(t,T)^2. \tag{31}$$

The zero-coupon rate is the affine function of the short rate $R(t,T) = \frac{A(t,T)r(t)+C(t,T)}{T-t}$, *and the long rate is* $R_\infty = \lim_{T\to\infty} R(t,T) = \theta - \frac{\sigma^2}{2k^2}$.

Proof. The first equality is the Laplace transform of a Gaussian variable (Proposition 17). Write $\tau = T-t$ and $A = A(t,T)$. Since $e^{-k\tau} = 1-kA$, $2k\tau - 3 + 4e^{-k\tau} - e^{-2k\tau} = 2k(\tau - A) - k^2A^2$, so that $\frac{1}{2}V = \frac{\sigma^2}{2k^2}(\tau - A) - \frac{\sigma^2}{4k}A^2$. Then $\bar{\xi} - \frac{1}{2}V = Ar(t) + \theta(\tau - A) - \frac{\sigma^2}{2k^2}(\tau - A) + \frac{\sigma^2}{4k}A^2 = Ar(t) + C$. When $\tau \to \infty$, $A \to 1/k$ and $C/\tau \to \theta - \sigma^2/(2k^2)$. □

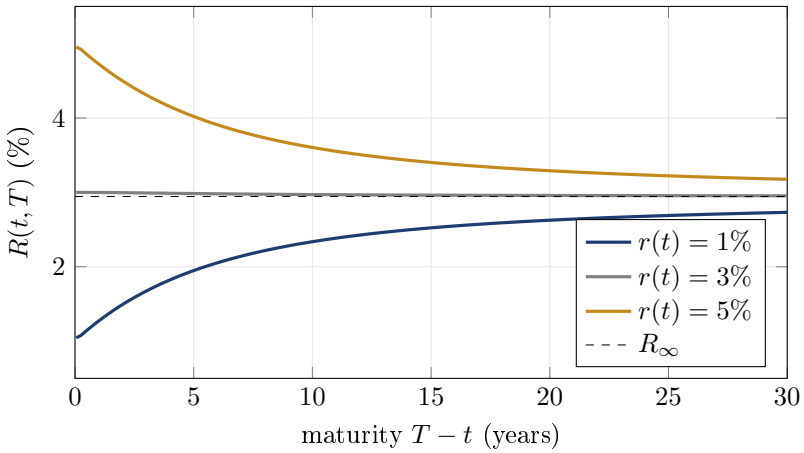


Figure 4: Zero-coupon curves in the Vasicek model ($k = 0.3$, $\theta = 3\%$, $\sigma = 1\%$)

Comments.

- The whole curve is a function of the single state variable $r(t)$, and its shape depends on it (Figure 4): increasing when $r(t)$ is low, decreasing when $r(t)$ is high. Slightly humped curves are also possible for intermediate values. For instance, with $r(t) = 1\%$, the 1-, 5-, 10- and 30-year rates are 1.27%, 1.95%, 2.34% and 2.73%.
- When the short rate moves by dr, the zero-coupon rate of maturity T moves by $\frac{A(t,T)}{T-t}dr$. This sensitivity decreases from 1 at the short end to 0 at the long end: short rates are more volatile than long rates.
- All rates are driven by the same Brownian motion: the instantaneous correlation between the variations of any two rates is equal to 1 (Exercise 4.3). The curve can shift but not twist. This motivates multi-factor models (Section 4.8).
- The long rate R_∞ is below θ: by Jensen's inequality, the convexity of $x \mapsto e^{-x}$ makes bonds more valuable, and rates lower, than with a deterministic short rate.

4.3 Route 2: the bond pricing PDE

The bond price is a function of the current short rate: $P(t,T) = p(t, r(t))$. By Proposition 4, $D(0,t)p(t,r(t))$ is a $\mathbb{Q}$-martingale, so its drift $D(0,t)\left(\partial_t p + k(\theta - r)\partial_r p + \frac{1}{2}\sigma^2 \partial_{rr} p - rp\right)$ vanishes, and p solves

$$\frac{\partial p}{\partial t} + k(\theta - r)\frac{\partial p}{\partial r} + \frac{1}{2}\sigma^2 \frac{\partial^2 p}{\partial r^2} - rp = 0, \qquad p(T, r) = 1. \tag{32}$$

Conversely, by the Feynman-Kac theorem (Theorem 4), the solution of (32) is the expectation (12).

Such a PDE is generally hard to solve, but the structure of the solution can be guessed. By (24), the derivative $\partial r(u)/\partial r(t) = e^{-k(u-t)}$ is deterministic, so that differentiating $p = \mathbb{E}\left[e^{-\int_t^T r(u)du} \mid r(t)\right]$ under the expectation gives

$$\partial_r p = -\left(\int_t^T e^{-k(u-t)} du\right) p = -A(t,T)p.$$

Hence $p(t,r) = \exp\left(-A(t,T)r - C(t,T)\right)$ for some function C. With $\partial_t A = kA - 1$, the PDE becomes

$$-(kA-1)r - \partial_t C - k(\theta - r)A + \frac{1}{2}\sigma^2 A^2 - r = 0 \quad \Longleftrightarrow \quad \partial_t C = -k\theta A + \frac{1}{2}\sigma^2 A^2,$$

the terms in r cancelling. With $C(T,T) = 0$,

$$C(t,T) = \int_t^T \left(k\theta A(u,T) - \frac{1}{2}\sigma^2 A(u,T)^2\right) du = \theta\left(T - t - A(t,T)\right) - \frac{1}{2}V(t,T),$$

since $\int_t^T kA(u,T)\,du = T - t - A(t,T)$ and $\sigma^2 \int_t^T A(u,T)^2 du = V(t,T)$ (compare with the computation of V above). We recover Proposition 10.

4.4 Route 3: the forward measure

By Proposition 7, $f(t,T) = \mathbb{E}^T\left[r(T) \mid \mathcal{F}_t\right]$: the forward rate is the expected short rate under the T-forward measure. The density process of $\mathbb{Q}^T$ is $Z_t = D(0,t)P(t,T)/P(0,T)$, a $\mathbb{Q}$-martingale whose Brownian part comes from $P(t,T) = e^{-A(t,T)r(t) - C(t,T)}$: $dZ_t = -Z_t \sigma A(t,T)\,dW_t$. By Corollary 1, $W_t^T = W_t + \int_0^t \sigma A(s,T)\,ds$ is a $\mathbb{Q}^T$-Brownian motion, and

$$dr(t) = \left[k\left(\theta - r(t)\right) - \sigma^2 A(t,T)\right] dt + \sigma\, dW_t^T \quad \text{under } \mathbb{Q}^T. \tag{33}$$

The function $m(u) = \mathbb{E}^T\left[r(u) \mid \mathcal{F}_t\right]$, $u \in [t,T]$, solves the linear ODE $m'(u) = k\left(\theta - m(u)\right) - \sigma^2 A(u,T)$ with $m(t) = r(t)$. Its value at $u = T$ is $r(t)e^{-k(T-t)} + \int_t^T e^{-k(T-u)}\left(k\theta - \sigma^2 A(u,T)\right) du$, that is

$$f(t,T) = \theta + \left(r(t) - \theta\right) e^{-k(T-t)} - \frac{\sigma^2}{2k^2}\left(1 - e^{-k(T-t)}\right)^2. \tag{34}$$

The forward rate is the risk-neutral expectation (25) of the future short rate, minus a convexity term which increases with the maturity. Integrating in T, $\int_t^T f(t,u)\,du = \bar{\xi}(t,T) - \frac{1}{2}V(t,T)$, which gives back the bond price $P(t,T) = \exp\left(-\int_t^T f(t,u)\,du\right)$ a third time.

4.5 Options on bonds, caps and swaptions

Proposition 11. *In the Vasicek model, for $t \le T < S$, the European call of maturity T and strike K on the zero-coupon bond of maturity S is worth*

$$ZBC(t,T,S,K) = P(t,S)N(h) - KP(t,T)N(h-\sigma_p),$$
$$ZBP(t,T,S,K) = KP(t,T)N(-h+\sigma_p) - P(t,S)N(-h) \quad \textit{(put)},$$

with $h = \frac{1}{\sigma_p}\ln\frac{P(t,S)}{KP(t,T)} + \frac{\sigma_p}{2}$ *and* $\sigma_p = \sigma A(T,S)\sqrt{\frac{1-e^{-2k(T-t)}}{2k}}$.

Proof. The call pays $(P(T,S)-K)^+$ at T. The forward price $F_u = P(u,S)/P(u,T)$ is a $\mathbb{Q}^T$-martingale with $F_T = P(T,S)$ (Proposition 7). Since $\ln F_u = -\left(A(u,S)-A(u,T)\right)r(u) + \text{deterministic}$ and $A(u,S)-A(u,T) = e^{-k(T-u)}A(T,S)$, its volatility $\sigma e^{-k(T-u)}A(T,S)$ is deterministic. Black's formula (Proposition 8) applies with $v^2 = \sigma^2 A(T,S)^2\int_t^T e^{-2k(T-u)}du = \sigma_p^2$. □

Caps. By Proposition 2, a caplet is $N(1+\tau_i K)$ puts $ZBP\left(t, T_{i-1}, T_i, \frac{1}{1+\tau_i K}\right)$: caps have closed-form prices in the Vasicek model. With $k = 0.3$, $\theta = 3\%$, $\sigma = 1\%$ and $r_0 = 3\%$, the caplet on the six-month rate fixing in one year with strike 3% is worth 0.00160 per unit of notional, which corresponds to a Black volatility of 27.3% (Exercise 4.5).

Swaptions. A receiver swaption is a put on a coupon bond (8), that is an option on a *sum* of zero-coupon bonds. In a one-factor model, this sum can nevertheless be decomposed.

Proposition 12 (Jamshidian's decomposition)**.** *Assume that $P(T_\alpha,T_i) = p_i\left(r(T_\alpha)\right)$ with p_i decreasing, as in the Vasicek model. Let r^* solve $\sum_{i=\alpha+1}^{\beta} c_i p_i(r^*) = 1$ and $K_i = p_i(r^*)$. Then*

$$\left(\sum_{i=\alpha+1}^{\beta} c_i P(T_\alpha,T_i) - 1\right)^+ = \sum_{i=\alpha+1}^{\beta} c_i\left(P(T_\alpha,T_i)-K_i\right)^+,$$

so that the receiver and the payer swaptions are respectively worth

$$N\sum_{i=\alpha+1}^{\beta} c_i\, ZBC(t,T_\alpha,T_i,K_i) \quad \textit{and} \quad N\sum_{i=\alpha+1}^{\beta} c_i\, ZBP(t,T_\alpha,T_i,K_i).$$

Proof. If $r(T_\alpha) \le r^*$, all the $P(T_\alpha,T_i)$ are above their K_i, and $\sum_i c_i P(T_\alpha,T_i) - 1 = \sum_i c_i\left(P(T_\alpha,T_i)-K_i\right) \ge 0$. If $r(T_\alpha) > r^*$, all the terms are below their K_i and both sides vanish. The payer case is identical with $\left(1-\sum_i c_i P_i\right)^+ = \sum_i c_i\left(K_i - P_i\right)^+$. □

The decomposition works because, in a one-factor model, all bond prices move in the same direction: this is the same feature which makes all rates perfectly correlated.

4.6 Fitting the initial curve: Hull-White and Ho-Lee

With three parameters, the Vasicek model cannot reproduce the observed discount curve $T \mapsto P^M(0,T)$: the prices of the bonds and swaps used to hedge would be wrong. The **Hull-White** model adds a deterministic function of time:

$$r(t) = x(t) + \varphi(t), \qquad dx(t) = -ax(t)\,dt + \sigma\,dW_t, \quad x(0) = 0,$$

equivalently $dr(t) = \left(\vartheta(t) - ar(t)\right)dt + \sigma\,dW_t$ with $\vartheta(t) = \varphi'(t) + a\varphi(t)$.

Proposition 13. *Let A and V be the functions of the Vasicek model with $k = a$. Then*

$$P(t,T) = \exp\left(-\int_t^T \varphi(u)\,du - A(t,T)x(t) + \frac{1}{2}V(t,T)\right).$$

The model reproduces the market curve, $P(0,T) = P^M(0,T)$ for all T, if and only if

$$\varphi(T) = f^M(0,T) + \frac{\sigma^2}{2a^2}\left(1-e^{-aT}\right)^2, \tag{35}$$

where $f^M(0,T) = -\partial_T \ln P^M(0,T)$ is the market instantaneous forward rate. In that case,

$$P(t,T) = \frac{P^M(0,T)}{P^M(0,t)}\exp\left(-A(t,T)x(t) + \frac{1}{2}\left[V(t,T) - V(0,T) + V(0,t)\right]\right).$$

Proof. x is a Vasicek process with $\theta = 0$, so $\int_t^T x(u)\,du$ is Gaussian with mean $A(t,T)x(t)$ and variance $V(t,T)$, and $\int_t^T \varphi$ is deterministic. At $t = 0$, $-\ln P^M(0,T) = \int_0^T \varphi - \frac{1}{2}V(0,T)$; differentiating in T gives $\varphi(T) = f^M(0,T) + \frac{1}{2}\partial_T V(0,T)$ with $\partial_T V(0,T) = \frac{\sigma^2}{a^2}\left(1 - e^{-aT}\right)^2$. Finally, $\int_t^T \varphi = \int_0^T \varphi - \int_0^t \varphi$ is expressed with the market discount factors. □

The volatility of forward bond prices only depends on a and σ, so Proposition 11 and Jamshidian's decomposition hold unchanged. In practice, φ is given by today's curve and (a, σ) are calibrated to cap or swaption prices.

The **Ho-Lee** model is the limit $a \to 0$: $dr(t) = \vartheta(t)\,dt + \sigma\,dW_t$. Then $A(t,T) = T - t$, $V(t,T) = \sigma^2(T-t)^3/3$ and $\varphi(T) = f^M(0,T) + \sigma^2 T^2/2$. Without mean reversion, the variance of the short rate grows linearly with time, which is unrealistic for long maturities.

4.7 The Cox-Ingersoll-Ross model

To avoid negative rates, [3] make the volatility vanish at zero:

$$dr(t) = k\left(\theta - r(t)\right)dt + \sigma\sqrt{r(t)}\,dW_t. \tag{36}$$

The short rate remains non-negative, and positive when $2k\theta \geq \sigma^2$ (Feller condition). Its conditional law is a scaled non-central χ^2, with the same mean (25) as in the Vasicek model and variance $r(t)\frac{\sigma^2}{k}\left(e^{-k\tau} - e^{-2k\tau}\right) + \frac{\theta\sigma^2}{2k}\left(1 - e^{-k\tau}\right)^2$, $\tau = T - t$.

Proposition 14. *In the CIR model,* $P(t,T) = \exp\left(-A(\tau)r(t) - C(\tau)\right)$ *with* $\tau = T - t$, $h = \sqrt{k^2 + 2\sigma^2}$ *and*

$$A(\tau) = \frac{2\left(e^{h\tau} - 1\right)}{2h + (k+h)\left(e^{h\tau} - 1\right)}, \qquad C(\tau) = -\frac{2k\theta}{\sigma^2}\ln\frac{2he^{(k+h)\tau/2}}{2h + (k+h)\left(e^{h\tau} - 1\right)}.$$

Proof. The bond pricing PDE is $\partial_t p + k(\theta - r)\partial_r p + \frac{1}{2}\sigma^2 r\,\partial_{rr} p - rp = 0$ with $p(T,r) = 1$. Looking for $p = e^{-A(\tau)r - C(\tau)}$ and separating the terms in r gives the Riccati equation $A' = 1 - kA - \frac{1}{2}\sigma^2 A^2$ and $C' = k\theta A$, with $A(0) = C(0) = 0$, whose solution is the one above (Exercise 4.7). □

As in the Vasicek model, zero-coupon rates are affine in $r(t)$ and perfectly correlated. More generally, whenever the drift and the squared volatility of the short rate are affine in r, bond prices are exponential-affine with coefficients solving Riccati equations (affine term structure models). The CIR model can also be shifted by a deterministic function to fit the initial curve, and options on bonds have closed forms involving the non-central χ^2 distribution [2]. It cannot, however, produce negative rates.

4.8 Multi-factor Gaussian models

One-factor models make all rates perfectly correlated, which prevents the curve from twisting and misprices products depending on the correlation between rates, such as swaptions. The n-factor Gaussian model, called G2++ in [2] when $n = 2$ (it is equivalent to the two-factor Hull-White model), keeps the analytical tractability of the Vasicek model:

$$r(t) = \sum_{i=1}^n x_i(t) + \varphi(t), \qquad dx_i(t) = -a_i x_i(t)\,dt + \sigma_i\,dW_i(t), \quad x_i(0) = 0,$$

with $d\langle W_i, W_j\rangle_t = \rho_{ij}\,dt$ ($\rho_{ii} = 1$) and φ deterministic with $\varphi(0) = r(0)$. In the Vasicek model, $n = 1$ and $\varphi(t) = r_0 e^{-kt} + \theta\left(1 - e^{-kt}\right)$. As before,

$$x_i(T) = x_i(t)e^{-a_i(T-t)} + \sigma_i\int_t^T e^{-a_i(T-u)}dW_i(u), \qquad \mathrm{Cov}\left(x_i(T), x_j(T) \mid \mathcal{F}_t\right) = \frac{\rho_{ij}\sigma_i\sigma_j}{a_i + a_j}\left(1 - e^{-(a_i+a_j)(T-t)}\right).$$

Lemma 1. *Let* $I_i(t,T) = \int_t^T x_i(u)\,du$, $\tau = T - t$, $M_i(t,T) = \frac{1 - e^{-a_i\tau}}{a_i}$ *and* $M_{ij}(t,T) = \frac{1 - e^{-(a_i+a_j)\tau}}{a_i + a_j}$. *Conditionally on* $\mathcal{F}_t$, *the* $I_i(t,T)$ *are jointly Gaussian with* $\mathbb{E}\left[I_i(t,T) \mid \mathcal{F}_t\right] = M_i(t,T)x_i(t)$ *and*

$$\mathrm{Cov}\left(I_i(t,T), I_j(t,T) \mid \mathcal{F}_t\right) = \frac{\rho_{ij}\sigma_i\sigma_j}{a_i a_j}\left[\tau - M_i(t,T) - M_j(t,T) + M_{ij}(t,T)\right]. \tag{37}$$

In particular $\mathrm{Var}\left(I_i(t,T) \mid \mathcal{F}_t\right) = \frac{\sigma_i^2}{a_i^2}\left[\tau + \frac{2}{a_i}e^{-a_i\tau} - \frac{1}{2a_i}e^{-2a_i\tau} - \frac{3}{2a_i}\right]$, *which is* (29) *with* $k = a_i$.

Proof. Integrating the expression of $x_i(u)$ and exchanging the integrals (stochastic Fubini theorem),

$$I_i(t,T) = M_i(t,T)x_i(t) + \sigma_i \int_t^T \left(\int_s^T e^{-a_i(u-s)} du \right) dW_i(s) = M_i(t,T)x_i(t) + \sigma_i \int_t^T M_i(s,T)\, dW_i(s).$$

By Itô's isometry, the covariance is $\rho_{ij}\sigma_i\sigma_j \int_t^T M_i(s,T)M_j(s,T)\, ds = \frac{\rho_{ij}\sigma_i\sigma_j}{a_i a_j} \int_0^\tau (1-e^{-a_i v})(1-e^{-a_j v})\, dv$, which gives (37). □

Theorem 3. *Let $V(t,T) = \sum_{i,j} \mathrm{Cov}\left(I_i(t,T), I_j(t,T) \mid \mathcal{F}_t\right)$. The zero-coupon bond is worth*

$$P(t,T) = \exp\left(-\int_t^T \varphi(u)\, du - \sum_{i=1}^n M_i(t,T)x_i(t) + \frac{1}{2}V(t,T) \right).$$

The model fits the market curve if and only if $\varphi(T) = f^M(0,T) + \frac{1}{2}\partial_T V(0,T)$, and then

$$P(t,T) = \frac{P^M(0,T)}{P^M(0,t)} \exp\left(-\sum_{i=1}^n M_i(t,T)x_i(t) + \frac{1}{2}\left[V(t,T) - V(0,T) + V(0,t)\right] \right).$$

The proof is the same as for Proposition 13.

Forward rates and decorrelation. Differentiating $\ln P(t,T)$,

$$f(t,T) = \varphi(T) + \sum_i e^{-a_i(T-t)} x_i(t) - \frac{1}{2}\partial_T V(t,T)$$

and

$$df(t,T) = \alpha(t,T)\, dt + \sum_{i=1}^n \sigma_i e^{-a_i(T-t)} dW_i(t), \qquad \alpha(t,T) = \sum_{i,j} \rho_{ij}\sigma_i\sigma_j e^{-a_i(T-t)} M_j(t,T).$$

The drift α is not free: it is imposed by the absence of arbitrage (this is the Heath-Jarrow-Morton drift condition). The instantaneous correlation between two forward rates is

$$\frac{\sum_{i,j} \rho_{ij}\sigma_i\sigma_j e^{-a_i(T_1-t)} e^{-a_j(T_2-t)}}{\sqrt{\sum_{i,j} \rho_{ij}\sigma_i\sigma_j e^{-(a_i+a_j)(T_1-t)} \sum_{i,j} \rho_{ij}\sigma_i\sigma_j e^{-(a_i+a_j)(T_2-t)}}},$$

equal to 1 when $n = 1$ but not in general. With two factors, $a_1 = 0.05$, $a_2 = 1$, $\sigma_1 = 0.8\%$, $\sigma_2 = 1\%$ and $\rho_{12} = -0.6$, the correlation between the 1-year and the 2-year forward rates is 0.95, and between the 1-year and the 10-year forward rates 0.88: a slowly mean-reverting factor drives the level of the curve, a fast one its slope. Bond options also have closed-form prices in this model [2].

Exercises

Exercise 4.1 *The Ornstein-Uhlenbeck process*

Consider the Vasicek dynamics (23).

(a) Using Itô's formula, check that $r(t) = r_0 e^{-kt} + \theta\left(1 - e^{-kt}\right) + \sigma \int_0^t e^{-k(t-u)} dW_u$ solves (23).

(b) Compute $\mathbb{E}\left[r(t)\right]$ and $\mathrm{Var}\left(r(t)\right)$.

(c) Discuss the behaviour of $r(t)$ for small and large t: what do small and large mean? Compare with the Brownian case $k = 0$.

(d) Recover the solution from the formula for linear SDEs of Section 7.

Exercise 4.2 *Integrated short rate and zero-coupon bonds*

(a) Prove (27) and (28).

(b) Show that $\mathrm{Var}\left(\xi(0,T)\right) = \int_0^T \int_0^T \gamma(0,u,v)\, du\, dv$ and derive (29).

(c) Deduce the price of the zero-coupon bond and the zero-coupon rate as functions of k, θ, σ, r_0 and T.

(d) Compute the limits of $R(t,T)$ when $T \to t$ and when $T \to \infty$.

Exercise 4.3 *Covariance structure of the Vasicek curve*

Let $\bar{A}(t,T) = A(t,T)/(T-t)$.

(a) Show that $d\langle R(\cdot,T_1), R(\cdot,T_2)\rangle_t = \sigma^2 \bar{A}(t,T_1)\bar{A}(t,T_2)\,dt$.

(b) Deduce that the instantaneous correlation between $dR(t,T_1)$ and $dR(t,T_2)$ is equal to 1. Comment on the covariance structure of the yield curve in a one-factor model.

Exercise 4.4 *Negative rates*

With $k = 0.3$, $\theta = 3\%$, $\sigma = 1\%$ and $r_0 = 1\%$, compute the probability that the short rate is negative in one year, and in the long run.

Exercise 4.5 *A caplet in the Vasicek model*

With $k = 0.3$, $\theta = 3\%$, $\sigma = 1\%$ and $r_0 = 3\%$, consider the caplet on the six-month rate fixing in one year ($T = 1$, $S = 1.5$, $\tau = 0.5$) with strike $K = 3\%$.

(a) Compute $P(0,1)$, $P(0,1.5)$ and the forward rate $F(0;1,1.5)$.

(b) Compute σ_p and the price of the caplet per unit of notional.

(c) Which volatility, in Black's formula for caplets (Section 5), gives the same price?

Exercise 4.6 *Hull-White model with n factors*

Consider the model of Section 4.8.

(a) Show that $x_i(t) = \sigma_i \int_0^t e^{-a_i(t-u)} dW_i(u)$, $\mathbb{E}[x_i(t) \mid \mathcal{F}_u] = x_i(u)e^{-a_i(t-u)}$, and compute $\mathrm{Cov}(x_i(t), x_j(t))$ and $\mathrm{Var}(r(t))$.

(b) Prove Lemma 1 and Theorem 3.

(c) Ho-Lee: for $n = 1$, show that when $a \to 0$, $M(t,T) \to T-t$, $V(t,T) \to \sigma^2(T-t)^3/3$ and $\varphi(T) \to f^M(0,T) + \sigma^2 T^2/2$.

Exercise 4.7 *The Cox-Ingersoll-Ross model*

(a) Show that $p = e^{-A(\tau)r - C(\tau)}$ solves the CIR bond pricing PDE if and only if $A' = 1 - kA - \frac{1}{2}\sigma^2 A^2$ and $C' = k\theta A$, with $A(0) = C(0) = 0$.

(b) Check that the function A of Proposition 14 solves the Riccati equation, and that it tends to the Vasicek function $(1 - e^{-k\tau})/k$ when $\sigma \to 0$.

(c) Is the Feller condition satisfied for $k = 0.5$, $\theta = 3\%$ and $\sigma = 10\%$?

5 Market models: the lognormal forward-LIBOR model

Short-rate models deduce the prices of caps and swaptions from the dynamics of an unobservable quantity, the short rate. The market, however, quotes caps and swaptions through Black's formula, as if each forward rate, or each swap rate, were lognormal. These formulas were long used without an underlying arbitrage-free dynamics of rates. *Market models* fill this gap: they model directly the observable forward rates, in such a way that Black's formula for caplets is exact. The lognormal forward-LIBOR model (LFM), also called BGM model after [1], is the reference model of this family.

5.1 Black's formula for caps

Consider the caplet paying $N\tau\,(L(T,S) - K)^+$ at S, with $\tau = \tau(T,S)$, and the forward rate $F(t) = F(t;T,S)$. By Proposition 7, F is a $\mathbb{Q}^S$-martingale and $F(T) = L(T,S)$, so that the caplet is worth

$$N\tau P(t,S)\,\mathbb{E}^S\left[(F(T) - K)^+ \mid \mathcal{F}_t\right].$$

Assume that $dF(t) = \sigma(t)F(t)\,dW_t^S$ under $\mathbb{Q}^S$, with σ deterministic, and let $\bar{\sigma}^2 = \frac{1}{T-t}\int_t^T \sigma(s)^2 ds$ be the average variance. By Proposition 8,

$$\text{Caplet}(t) = N\tau P(t,S)\ \text{Bl}\left(F(t), K, \bar{\sigma}\sqrt{T-t}\right), \tag{38}$$

and the floorlet is worth $N\tau P(t,S)\,(KN(-d_2) - F(t)N(-d_1))$. The uncertainty stops at the fixing date T, whereas the payment occurs at S: S is the *natural* payment date of the rate $L(T,S)$, the one for which the forward rate is a martingale under the payment measure. Paying the same rate at another date requires a convexity adjustment (Section 5.4).

Caps. The market quotes a cap on the schedule $\{T_\alpha, \dots, T_\beta\}$ through a single volatility $\sigma_{\alpha,\beta}$, used for all its caplets:

$$\text{Cap}(0) = N \sum_{i=\alpha+1}^{\beta} \tau_i P(0,T_i)\ \text{Bl}\left(F_i(0), K, \sigma_{\alpha,\beta}\sqrt{T_{i-1}}\right). \tag{39}$$

The maturity in Black's formula is the fixing date T_{i-1} of each caplet. The volatilities $\bar{\sigma}_i$ of the individual caplets are then stripped from the cap volatilities of increasing maturities. For instance, with $F = K = 3\%$, $\bar{\sigma} = 20\%$, a fixing in one year, $\tau = 0.5$, $P(0,1.5) = 0.955$ and $N = 10$ million, the caplet is worth 11,411.

Negative rates. Black's formula assumes $F > 0$. When euro rates became negative (2014–2022), the market moved to normal (Bachelier) volatilities: if $dF(t) = \sigma_N\, dW_t^S$,

$$\text{Caplet}(t) = N\tau P(t,S)\left[(F(t)-K)N(d) + \sigma_N\sqrt{T-t}\,n(d)\right], \qquad d = \frac{F(t)-K}{\sigma_N\sqrt{T-t}}. \tag{40}$$

Another solution is the shifted lognormal model, in which $F+\delta$ is lognormal for some shift $\delta > 0$: Black's formula then applies to $F+\delta$ and $K+\delta$.

5.2 The model

Black's formula cannot handle:

- payoffs depending on several forward rates, such as ratchets, spread options or swaptions: they require the *joint* dynamics of the rates;
- a rate paid at a date which is not its natural payment date, such as in-arrears payments: the rate is then needed under another measure.

One needs the dynamics of all the forward rates of a schedule under a single measure.

Let $T_0 < T_1 < \cdots < T_M$ be a schedule, $\tau_k = \tau(T_{k-1}, T_k)$, $F_k(t) = F(t; T_{k-1}, T_k)$ the k-th forward rate, alive until its fixing date T_{k-1}, and $\mathbb{Q}^k$ the T_k-forward measure, under which F_k is a martingale. The **LFM** assumes that, for each k,

$$dF_k(t) = \sigma_k(t)F_k(t)\,dZ_k(t) \quad \text{under } \mathbb{Q}^k, \qquad t \le T_{k-1}, \tag{41}$$

where σ_k is deterministic, $Z = (Z_1, \dots, Z_M)$ is a Brownian motion with correlations $d\langle Z_i, Z_j\rangle_t = \rho_{ij}\,dt$, and Z_k is a $\mathbb{Q}^k$-Brownian motion. Each caplet is then exactly priced by (38): the LFM is consistent with the market formula for caps.

Volatility structure. Let $\beta(t) = m$ when $T_{m-2} < t \le T_{m-1}$ (with $T_{-1} = 0$): $F_{\beta(t)}$ is the first forward rate which is not fixed yet. A common choice is a volatility which is constant on each period, $\sigma_k(t) = \sigma_{k,\beta(t)}$, parametrised as $\sigma_{k,j} = \phi_k\psi_{k-j+1}$ (Table 2): ψ describes how the volatility depends on the number of periods left before the fixing, and ϕ_k rescales it so that the caplet volatilities are matched: $T_{k-1}\bar{\sigma}_k^2 = \sum_{j=1}^{k}(T_{j-1}-T_{j-2})\sigma_{k,j}^2$.

	$t \in (0,T_0]$	$(T_0,T_1]$	$(T_1,T_2]$	$\cdots$	$(T_{M-2},T_{M-1}]$
$F_1(t)$	$\phi_1\psi_1$	fixed	fixed	$\cdots$	fixed
$F_2(t)$	$\phi_2\psi_2$	$\phi_2\psi_1$	fixed	$\cdots$	fixed
$F_3(t)$	$\phi_3\psi_3$	$\phi_3\psi_2$	$\phi_3\psi_1$	$\cdots$	fixed
$\vdots$				$\ddots$	
$F_M(t)$	$\phi_M\psi_M$	$\phi_M\psi_{M-1}$	$\phi_M\psi_{M-2}$	$\cdots$	$\phi_M\psi_1$

Table 2: Piecewise constant volatilities $\sigma_{k,\beta(t)} = \phi_k\psi_{k-\beta(t)+1}$ in the LFM

Correlations. The matrix (ρ_{ij}) does not affect caps. It is usually parametrised, for instance as

$$\rho_{ij} = \exp\left(-\lambda|T_i - T_j|\right), \qquad \lambda > 0,$$

so that rates of close maturities are more correlated, and calibrated to swaptions. The possible choices are discussed in Chapters 6 and 7 of [2].

5.3 Dynamics under a common measure

Proposition 15. *Under $\mathbb{Q}^i$, for $t \le \min(T_{k-1}, T_i)$, the forward rate F_k follows*

$$k < i: \quad dF_k(t) = -\sigma_k(t)F_k(t) \sum_{j=k+1}^{i} \frac{\rho_{kj}\tau_j\sigma_j(t)F_j(t)}{1+\tau_j F_j(t)}\, dt + \sigma_k(t)F_k(t)\, dZ_k^i(t),$$

$$k > i: \quad dF_k(t) = +\sigma_k(t)F_k(t) \sum_{j=i+1}^{k} \frac{\rho_{kj}\tau_j\sigma_j(t)F_j(t)}{1+\tau_j F_j(t)}\, dt + \sigma_k(t)F_k(t)\, dZ_k^i(t),$$

where Z_k^i is a $\mathbb{Q}^i$-Brownian motion.

Proof. Take $k < i$. By (13) with the numeraires $P(\cdot, T_k)$ and $P(\cdot, T_i)$,

$$H_t = \left.\frac{d\mathbb{Q}^i}{d\mathbb{Q}^k}\right|_{\mathcal{F}_t} = \frac{P(0,T_k)}{P(0,T_i)}\frac{P(t,T_i)}{P(t,T_k)} = \frac{P(0,T_k)}{P(0,T_i)} \prod_{j=k+1}^{i} \frac{1}{1+\tau_j F_j(t)}.$$

By Theorem 2, $Z_k^i = Z_k - \langle Z_k, \ln H\rangle$ is a $\mathbb{Q}^i$-Brownian motion, with

$$d\langle Z_k, \ln H\rangle_t = -\sum_{j=k+1}^{i} \frac{\tau_j\, d\langle Z_k, F_j\rangle_t}{1+\tau_j F_j(t)} = -\sum_{j=k+1}^{i} \frac{\rho_{kj}\tau_j\sigma_j(t)F_j(t)}{1+\tau_j F_j(t)}\, dt.$$

Substituting $dZ_k = dZ_k^i + d\langle Z_k, \ln H\rangle$ in (41) gives the first formula. For $k > i$, $H_t \propto \prod_{j=i+1}^{k}(1+\tau_j F_j(t))$ and the sign changes. □

The drift terms are covariances between F_k and the ratio of the two numeraires: they are of the same nature as quanto adjustments. They depend on the current forward rates, so that F_k is not lognormal under $\mathbb{Q}^i$ and closed-form prices are not available in general: expectations are computed by Monte Carlo simulation of the vector $(\ln F_k)_k$. A good approximation is often obtained by *freezing* the drift, that is by replacing $F_j(t)$ by $F_j(0)$ in it: the forward rates then remain lognormal under $\mathbb{Q}^i$, with a deterministic drift. The approximation is good as long as $\tau_j\sigma_j F_j$ remains small.

5.4 Two applications

Ratchets and spreads. A ratchet pays at T_i a function of the difference between two consecutive rates, for instance $\tau_i\left(L(T_{i-1},T_i) - L(T_{i-2},T_{i-1}) - X\right)^+$. Using the numeraire $P(\cdot,T_i)$, its price is

$$\tau_i P(0,T_i)\,\mathbb{E}^i\left[\left(F_i(T_{i-1}) - F_{i-1}(T_{i-2}) - X\right)^+\right].$$

Under $\mathbb{Q}^i$, F_i is a martingale and, by Proposition 15,

$$dF_{i-1}(t) = \sigma_{i-1}F_{i-1}(t)\left(-\frac{\rho_{i-1,i}\tau_i\sigma_i F_i(t)}{1+\tau_i F_i(t)}\, dt + dZ_{i-1}^i(t)\right).$$

With a frozen drift, both rates are lognormal and the ratchet becomes an option on the spread of two correlated lognormal variables. Margrabe's formula (Exercise 3.2) is exact only when $X = 0$ and both rates are observed at the same date; otherwise one uses an approximation (Kirk's formula) or a Monte Carlo simulation.

In-arrears payments and convexity adjustment. An in-arrears payment pays $\tau L(T,S)$ at the fixing date T instead of S. Receiving $\tau L(T,S)$ at T is equivalent to receiving $\tau L(T,S)/P(T,S) = \tau L(T,S)\left(1+\tau L(T,S)\right)$ at S. Using the numeraire $P(\cdot,S)$, the price is

$$P(t,S)\,\mathbb{E}^S\left[\tau F(T)\left(1+\tau F(T)\right) \mid \mathcal{F}_t\right] = \tau P(t,S)\left[F(t) + \tau\,\mathbb{E}^S\left[F(T)^2 \mid \mathcal{F}_t\right]\right].$$

When F is lognormal, $\mathbb{E}^S\left[F(T)^2 \mid \mathcal{F}_t\right] = F(t)^2 e^{v^2}$ with $v^2 = \int_t^T \sigma(s)^2 ds$. Dividing the price by $\tau P(t,T)$ gives the expected rate under the payment measure $\mathbb{Q}^T$:

$$\mathbb{E}^T\left[L(T,S) \mid \mathcal{F}_t\right] = F(t) + \frac{\tau F(t)^2\left(e^{v^2}-1\right)}{1+\tau F(t)}. \tag{42}$$

The second term is the **convexity adjustment**: an in-arrears rate is worth more than the forward rate (see also Exercise 3.4). With $F = 3\%$, $\sigma = 20\%$ and $\tau = 1$, it is 4.3 basis points for $T = 10$ years, but only 0.2 basis point for $T = 1$ and $\tau = 0.5$.

5.5 Swaptions in the LFM

A swaption depends on the joint law of the forward rates $F_{\alpha+1}, \ldots, F_\beta$ at T_α, which has no closed form in the LFM. By Proposition 1, $S_{\alpha,\beta} = \sum_i w_i F_i$, where the weights w_i vary little. Freezing them at their initial values, and freezing the forward rates in the volatility terms, the quadratic variation of $S_{\alpha,\beta}$ can be matched with that of a lognormal process. This gives **Rebonato's formula**:

$$\left(\sigma_{\alpha,\beta}^{\mathrm{LFM}}\right)^2 \approx \frac{1}{T_\alpha} \sum_{i,j=\alpha+1}^{\beta} \frac{w_i(0) w_j(0) F_i(0) F_j(0) \rho_{ij}}{S_{\alpha,\beta}(0)^2} \int_0^{T_\alpha} \sigma_i(t)\sigma_j(t)\, dt, \tag{43}$$

to be used in Black's formula for swaptions (19). The swaption volatility decreases when the correlations decrease: this is how the correlation matrix is calibrated to swaption prices, once the volatilities have been calibrated to caps.

Exercises

Exercise 5.1 *Pricing a caplet*

A caplet on a six-month rate ($\tau = 0.5$) fixing in one year has a notional of 10 million and a strike $K = 3\%$. The forward rate is $F = 3\%$, the caplet volatility $\bar{\sigma} = 20\%$ and $P(0, 1.5) = 0.955$.

(a) Compute the price of the caplet.

(b) Deduce the price of the floorlet with the same strike.

(c) Why is the maturity in Black's formula the fixing date and not the payment date or the accrual period τ?

Exercise 5.2 *Normal volatilities*

Assume that $dF(t) = \sigma_N\, dW_t^S$ under $\mathbb{Q}^S$.

(a) Prove the Bachelier formula (40).

(b) Show that at the money, the caplet is worth $N\tau P(t,S)\sigma_N\sqrt{T-t}/\sqrt{2\pi}$, and that Black's formula at the money is approximately $N\tau P(t,S)F\bar{\sigma}\sqrt{T-t}/\sqrt{2\pi}$. Deduce the relation between normal and lognormal volatilities at the money.

Exercise 5.3 *Two forward rates*

Consider the schedule $T_0 = 1$, $T_1 = 2$, $T_2 = 3$ ($\tau_1 = \tau_2 = 1$) and the forward rates F_1 and F_2, with constant volatilities.

(a) Write the dynamics of F_1 under $\mathbb{Q}^2$.

(b) With $F_1(0) = F_2(0) = 3\%$, $\sigma_1 = \sigma_2 = 20\%$ and $\rho_{12} = 0.8$, compute the frozen drift and the approximate value of $\mathbb{E}^2\left[F_1(T_0)\right]$. Comment.

Exercise 5.4 *Convexity adjustment*

(a) Prove (42) and check that it is consistent with Exercise 3.4.

(b) Compute the convexity adjustment for $F = 3\%$, $\sigma = 20\%$, $\tau = 1$ and $T = 10$ years, then for $\sigma = 30\%$ and $T = 5$ years.

6 After LIBOR: compounded risk-free rates

6.1 Context

The LIBOR (London Interbank Offered Rate) was computed from the declarations of a panel of banks. After the manipulation scandals of the early 2010s, regulators organised the transition to overnight **risk-free rates** (RFR) computed from actual transactions: €STR for the euro, SOFR for the US dollar, SONIA for the pound sterling, SARON for the Swiss franc and TONA for the yen. Most LIBOR settings ceased at the end of 2021, the main US dollar ones after June 2023. The Euribor, whose methodology was reformed, is still published: as a forward-looking term rate, it remains covered by the LFM of Section 5.

Derivatives on RFR are discounted with the curve built from overnight index swaps (OIS), and their floating legs pay the overnight rate *compounded in arrears* over each period. The key difference with LIBOR is the timing of the fixing: $L(T_{k-1}, T_k)$ is known at the *beginning* of the period, the compounded RFR only at its *end*. In this section, we work with a single curve: the RFR is the short rate of the discount curve.

6.2 Compounded rate and accumulation factor

On the period $[T_{k-1}, T_k]$, of length Δ_k, the compounded rate R_k is defined from the daily fixings r_{t_j} and accruals δ_j:

$$1 + \Delta_k R_k = \prod_j \left(1 + r_{t_j}\delta_j\right) \approx \exp\left(\int_{T_{k-1}}^{T_k} r(s)\,ds\right) = \frac{B(T_k)}{B(T_{k-1})}. \tag{44}$$

We call $X_k = 1 + \Delta_k R_k$ the **accumulation factor**. For instance, over 91 days at a constant overnight rate of 2% (act/360), daily compounding gives an accrual of 0.506822%, continuous compounding 0.506836%, and simple interest 0.505556%: the continuous approximation is excellent.

Proposition 16. *For $t \le T_k$,*

$$\mathbb{E}^{T_k}\left[X_k \mid \mathcal{F}_t\right] = \begin{cases} \dfrac{P(t,T_{k-1})}{P(t,T_k)} & t \le T_{k-1}, \\ \dfrac{B(t)}{B(T_{k-1})}\dfrac{1}{P(t,T_k)} & T_{k-1} < t \le T_k. \end{cases}$$

*Consequently, the **forward compounded rate** $R_k(t) = \mathbb{E}^{T_k}\left[R_k \mid \mathcal{F}_t\right]$ is a $\mathbb{Q}^{T_k}$-martingale on the whole interval $[0, T_k]$, and for $t \le T_{k-1}$ it is equal to the simply compounded forward rate $F(t; T_{k-1}, T_k)$.*

Proof. By Proposition 6,

$$\mathbb{E}^{T_k}\left[X_k \mid \mathcal{F}_t\right] = \frac{1}{P(t,T_k)}\mathbb{E}^{\mathbb{Q}}\left[D(t,T_k)\frac{B(T_k)}{B(T_{k-1})} \mid \mathcal{F}_t\right] = \frac{1}{P(t,T_k)}\mathbb{E}^{\mathbb{Q}}\left[\frac{B(t)}{B(T_{k-1})} \mid \mathcal{F}_t\right].$$

For $t \le T_{k-1}$, the last expectation is $P(t,T_{k-1})$; for $t > T_{k-1}$, $B(t)/B(T_{k-1})$ is known at t. The forward rate $R_k(t) = \left(\mathbb{E}^{T_k}[X_k \mid \mathcal{F}_t] - 1\right)/\Delta_k$ is a conditional expectation, hence a martingale, and (2) gives the last statement. □

In a single-curve world, an RFR swap is therefore valued exactly like the IRS of Section 1: the formulas (3) and (4) still hold.

6.3 Caplets on compounded rates

A caplet on the compounded rate pays at T_k

$$N\Delta_k\left(R_k - K\right)^+ = N\left(X_k - (1 + \Delta_k K)\right)^+,$$

a call on the accumulation factor with strike $1 + \Delta_k K$. Using the numeraire $P(\cdot, T_k)$, it is worth

$$N\Delta_k P(t,T_k)\,\mathbb{E}^{T_k}\left[\left(R_k(T_k) - K\right)^+ \mid \mathcal{F}_t\right].$$

Unlike the forward LIBOR rate, which stops moving at T_{k-1}, the martingale $R_k(t)$ keeps moving during the accrual period, until the last fixing. The **forward market model** of [5] takes this into account with

$$dR_k(t) = \sigma_k g_k(t) R_k(t)\,dW_t^{T_k}, \qquad g_k(t) = \begin{cases} 1 & t \le T_{k-1}, \\ \frac{T_k - t}{T_k - T_{k-1}} & T_{k-1} < t \le T_k, \end{cases}$$

the volatility decreasing linearly to zero as the period is fixed day after day. The integrated variance seen from $t \le T_{k-1}$ is $\sigma_k^2\left[(T_{k-1} - t) + \Delta_k/3\right]$, hence

$$\text{Caplet}(t) = N\Delta_k P(t,T_k)\ \text{Bl}\left(R_k(t), K, \sigma_k\sqrt{T_{k-1}-t+\Delta_k/3}\right), \qquad t \le T_{k-1}. \tag{45}$$

Inside the accrual period, $T_{k-1} < t < T_k$, the remaining variance is $\sigma_k^2(T_k-t)^3/(3\Delta_k^2)$. Compared with a forward-looking rate fixing at T_{k-1}, the effective volatility is multiplied by $\sqrt{1+\Delta_k/(3(T_{k-1}-t))}$: 1.041 for a three-month period starting in one year.

Remarks.

- Since $X_k = 1+\Delta_k R_k$, assuming that the accumulation factor is lognormal and applying Black's formula to X_k with the strike $1+\Delta_k K$ amounts to a shifted lognormal model for R_k, with a shift $1/\Delta_k$. Both are market conventions; in practice, as for LIBOR caps since 2014, normal volatilities are widely used.
- The forward market model extends the LFM: it describes the joint dynamics of all the rates of a schedule under a common measure, with drifts similar to those of Proposition 15, and covers both backward-looking (RFR) and forward-looking (Euribor) rates.

Exercises

Exercise 6.1 *Forward compounded rates*

(a) Prove Proposition 16.

(b) Deduce the value of a swap exchanging a fixed rate K against the compounded RFR on a schedule $\{T_\alpha, \dots, T_\beta\}$.

Exercise 6.2 *Daily compounding*

The overnight rate is constant and equal to 2% (act/360) over a period of 91 days. Compute the accrual ΔR with daily compounding, with continuous compounding and with simple interest, and comment.

Exercise 6.3 *Volatility of a backward-looking rate*

(a) Show that $\int_t^{T_k} \sigma_k^2 g_k(s)^2 ds = \sigma_k^2\left[(T_{k-1}-t)+\Delta_k/3\right]$ for $t \le T_{k-1}$, and $\sigma_k^2(T_k-t)^3/(3\Delta_k^2)$ for $T_{k-1} < t \le T_k$.

(b) With $R_k(0) = K = 2\%$, $\sigma_k = 30\%$, $T_{k-1} = 1$ and $\Delta_k = 0.25$, compare $\text{Bl}(R_k(0), K, v)$ for a forward-looking rate fixing at T_{k-1} and for the compounded RFR.

7 Appendix

7.1 Laplace transform of a Gaussian variable

Proposition 17. *If $Y \sim \mathcal{N}(\mu, s^2)$, then for all $\lambda \in \mathbb{R}$, $\mathbb{E}\left(e^{\lambda Y}\right) = e^{\lambda\mu + \frac{1}{2}\lambda^2 s^2}$.*

7.2 Linear stochastic differential equations

The following result is Problem 5.6.15 of [4]. Consider the linear SDE

$$dX_t = [A(t)X_t + a(t)]\,dt + [\sigma_1(t)X_t + \sigma_2(t)]\,dW_t, \qquad X_0 \text{ given}, \tag{46}$$

where A, a, σ_1, σ_2 are bounded measurable functions of time. Define

$$Z_t = \exp\left(\int_0^t \sigma_1(s)\,dW_s - \frac{1}{2}\int_0^t \sigma_1^2(s)\,ds + \int_0^t A(s)\,ds\right).$$

Then the unique solution of (46) is

$$X_t = Z_t\left[X_0 + \int_0^t \frac{a(u)-\sigma_1(u)\sigma_2(u)}{Z_u}\,du + \int_0^t \frac{\sigma_2(u)}{Z_u}\,dW_u\right].$$

The verification by Itô's formula is left as an exercise. The Vasicek model corresponds to $A = -k$, $a = k\theta$, $\sigma_1 = 0$ and $\sigma_2 = \sigma$, the Black-Scholes model to $A = \mu$, $a = 0$, $\sigma_1 = \sigma$ and $\sigma_2 = 0$.

7.3 The Feynman-Kac theorem

Let X be the diffusion

$$dX_t = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dW_t,$$

with μ and σ Lipschitz in x, so that the SDE has a unique strong solution. As in [FP], define for $f \in C^{1,2}$

$$\mathcal{L}f(t,x) = \frac{\partial f}{\partial t} + \mu(t,x)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 f}{\partial x^2},$$

so that Itô's formula reads $df(t, X_t) = \mathcal{L}f(t, X_t)\,dt + \frac{\partial f}{\partial x}(t, X_t)\sigma(t, X_t)\,dW_t$.

Theorem 4 (Feynman-Kac)**.** *Let $\rho(t,x)$ be a function bounded from below and F a payoff function. Under suitable growth conditions, the solution f of the Cauchy (terminal value) problem*

$$\mathcal{L}f - \rho f = 0, \qquad f(T,x) = F(x),$$

is given by

$$f(t,x) = \mathbb{E}\left[\exp\left(-\int_t^T \rho(s, X_s)\,ds\right) F(X_T) \mid X_t = x\right].$$

Proof. Let $M_s = \exp\left(-\int_t^s \rho(u, X_u)\,du\right) f(s, X_s)$ for $s \in [t, T]$. By Itô's formula,

$$dM_s = e^{-\int_t^s \rho(u,X_u)du}\left[\left(\mathcal{L}f - \rho f\right)(s, X_s)\,ds + \frac{\partial f}{\partial x}(s, X_s)\sigma(s, X_s)\,dW_s\right].$$

If f solves the PDE, the drift vanishes and M is a local martingale, a true martingale under the growth conditions. Then $M_t = \mathbb{E}[M_T \mid \mathcal{F}_t]$, which is the formula since $M_t = f(t, X_t)$ and $M_T = e^{-\int_t^T \rho} F(X_T)$. Conversely, if f is defined by the expectation and is $C^{1,2}$, M is a martingale by the tower property and its drift must vanish. □

Three applications are used in these notes and in [FP]:

- $\rho = 0$: $\mathbb{E}[F(X_T) \mid X_t = x]$ solves $\mathcal{L}f = 0$;
- $\rho = r$ constant and X a stock under $\mathbb{Q}$: the Black-Scholes PDE;
- $\rho(t,x) = x$ and $X = r$ the short rate: the bond pricing PDE (32).

Solutions

Solutions to the Exercises

Solution 1.1

(a) $10^6 \times 2.5\% \times 181/360 = 12{,}569.44$ EUR. (b) The same interest must be obtained with $\tau = 181/365$: $2.5\% \times 365/360 = 2.5347\%$. (c) $P = 1/(1 + 2.5\% \times 181/360) = 0.987587$ and $R = -\ln P/(181/365) = 2.5189\%$.

Solution 1.2

(a) $F_1 = 1/0.97 - 1 = 3.093\%$, $F_2 = 0.97/0.94 - 1 = 3.191\%$, $F_3 = 0.94/0.91 - 1 = 3.297\%$. (b) $C = 0.97 + 0.94 + 0.91 = 2.82$ and $S = (1 - 0.91)/2.82 = 3.191\%$. (c) The weights are $0.97/2.82 = 0.3440$, $0.94/2.82 = 0.3333$ and $0.91/2.82 = 0.3227$, and $\sum_i w_i F_i = 3.191\%$. Note that $\tau_i P(0,T_i) F_i = P(0,T_{i-1}) - P(0,T_i) = 0.03$ for each i. (d) $RFS = 100\,(3\% \times 2.82 - 0.09) = -0.54$: the fixed rate is below the swap rate, receiving it has a negative value. (e) The weights are positive and sum to one, so S is a convex combination of the F_i. It is equal to them when the forward curve is flat.

Solution 1.3

(a) With $P = P(T_{i-1}, T_i)$ and $L = L(T_{i-1}, T_i)$, $1 + \tau_i L = 1/P$, so $\tau_i P(L - K) = 1 - P - \tau_i K P = (1 + \tau_i K)\left(\frac{1}{1+\tau_i K} - P\right)$. Since $P > 0$, taking positive parts gives (6). The left-hand side is the value at T_{i-1} of the caplet payoff, which is known at T_{i-1} and paid at T_i. (b) Caplet by caplet, $(L - K)^+ - (K - L)^+ = L - K$: the difference pays $N\tau_i(L - K)$ at each T_i, which is the payer swap. (c) Same argument at T_α with

$x = C_{\alpha,\beta}(T_\alpha)(S_{\alpha,\beta}(T_\alpha) - K)$: the difference pays the value at T_α of the payer swap, whose value at t is $NC_{\alpha,\beta}(t)\,(S_{\alpha,\beta}(t) - K)$ by (3). (d) By Proposition 7, the caplet is worth at T_α

$$N\tau_i P(T_\alpha, T_i)\mathbb{E}^{T_i}\left[(L(T_{i-1}, T_i) - K)^+ \mid \mathcal{F}_{T_\alpha}\right],$$

and $\mathbb{E}^{T_i}\left[L(T_{i-1}, T_i) \mid \mathcal{F}_{T_\alpha}\right] = F_i(T_\alpha)$ since F_i is a $\mathbb{Q}^{T_i}$-martingale. Jensen's inequality for the convex function $x \mapsto (x - K)^+$ gives the bound. Summing over i and using the inequality $\left(\sum x_i\right)^+ \leq \sum x_i^+$, the value of the cap at T_α is larger than the payoff of the swaption, hence the result at any $t \leq T_\alpha$.

Solution 1.4

(a) $\ln P(0,T) = -aT - \frac{b}{2}T^2$ so $f(0,T) = a + bT$, which is above $R(0,T)$ for $T > 0$ since the curve is upward sloping. (b) $P(0,1) = e^{-0.022}$ and $P(0,2) = e^{-0.048}$, so $F(0;1,2) = e^{0.026} - 1 = 2.634\%$, close to $f(0,1.5) = 2.6\%$: the difference comes from simple versus continuous compounding.

Solution 2.1

(a) $D(t,T)$ is deterministic, so it goes out of the expectation in (12). Then $\ln P(t,T) = -\int_t^T r(s)\,ds$ and $f(t,T) = -\partial_T \ln P(t,T) = r(T)$. (b) $P(t,T)/P(t,S) = \exp\left(\int_T^S r(s)\,ds\right) = 1/P(T,S)$, so that $F(t;T,S) = \frac{1}{\tau}\left(\frac{1}{P(T,S)} - 1\right) = L(T,S)$.

Solution 2.2

(a) A stock paying a continuous dividend yield q: the traded asset is the stock with dividends reinvested, worth $S_t e^{qt}$. It must drift at r, hence $c = r - q$. An exchange rate X (domestic price of one unit of foreign currency): the traded asset is the foreign currency invested in the foreign bank account, worth $X_t B^f(t)$ in domestic currency, hence $c = r - r^f$ (Section 3). A commodity: holding it costs storage u and brings a convenience yield y, hence $c = r + u - y$. (b) With a constant rate, $P(0,T) = e^{-rT}$ and the forward measure is $\mathbb{Q}$. The forward price is $\mathbb{E}^{\mathbb{Q}}(S_T) = S_0 e^{cT}$, which is also the cost of the cash and carry strategy including its carry.

Solution 2.3

(a) $P(0,1) = 1/1.02 = 0.980392$ and $P(0,2) = \frac{1}{1.02}\left(\frac{0.5}{1.01} + \frac{0.5}{1.03}\right) = 0.961261$, slightly above $1/1.02^2 = 0.961169$: by Jensen's inequality, $\mathbb{E}\left(\frac{1}{1+r_1}\right) > \frac{1}{1+\mathbb{E}(r_1)}$. (b) $F(0;1,2) = P(0,1)/P(0,2) - 1 = 1.990\%$, below $\mathbb{E}^{\mathbb{Q}}(r_1) = 2\%$: the forward rate is not the risk-neutral expectation of the future rate. (c) Under $\mathbb{Q}^2$, the probability of the state $r_1 = 3\%$ is $\frac{1/1.03}{1/1.01 + 1/1.03} = 0.4951$, so that $\mathbb{E}^{\mathbb{Q}^2}(r_1) = 0.5049 \times 1\% + 0.4951 \times 3\% = 1.990\% = F(0;1,2)$. This is a first example of Proposition 7: forward rates are expectations under forward measures.

Solution 3.1

(a) $S_T = S_0 \exp\left((c - \frac{1}{2}\sigma^2)T + \sigma W_T\right)$, so $F_T = S_0 e^{cT}$ and $S_T = F_T Y_T$. (b) $S_T \geq K \Leftrightarrow W_T/\sqrt{T} \geq -d_A$, and $W_T/\sqrt{T} \sim \mathcal{N}(0,1)$, hence $\mathbb{Q}(S_T \geq K) = N(d_A)$. (c) $Y_T > 0$ and $\mathbb{E}^{\mathbb{Q}}(Y_T) = 1$. The density process is Y_t, with $dY_t = Y_t \sigma\, dW_t$, so $W_t - \sigma t$ is a $\mathbb{Q}^S$-Brownian motion by Corollary 1. When $c = r$, $Y_t = S_t e^{-rt}/S_0$: by (13), $\mathbb{Q}^S$ is the measure of the numeraire S. (d) $B = F_T \mathbb{E}^{\mathbb{Q}}\left[Y_T \mathbf{1}_{S_T \geq K}\right] = F_T \mathbb{Q}^S(S_T \geq K)$. Under $\mathbb{Q}^S$, $\ln S_T = \ln F_T + \frac{1}{2}\sigma^2 T + \sigma W_T^S$, hence $\mathbb{Q}^S(S_T \geq K) = N(d_B)$ with $d_B = \frac{\ln(F_T/K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$. (e) $C = e^{-rT}\left(F_T N(d_B) - K N(d_A)\right)$. For $c = r$, $e^{-rT} F_T = S_0$ and $C = S_0 N(d_1) - K e^{-rT} N(d_2)$.

Solution 3.2

(a) Itô's formula with $d(1/y) = -dy/y^2 + d\langle y \rangle / y^3$, then the product rule, the cross term being $-\rho\sigma_X\sigma_Y\, dt$. (b) $Z_T = \frac{Y_T e^{-rT}}{Y_0}$ by (13), and $Z_t = Y_t e^{-rt}/Y_0$ satisfies $dZ_t = Z_t \sigma_Y\, dW_t^Y$. (c) Corollary 1 with $\lambda = \sigma_Y$ and the correlations ρ and 1. Substituting in (a), the drift vanishes: $d(X/Y) = (X/Y)\left(\sigma_X\, d\tilde{W}^X - \sigma_Y\, d\tilde{W}^Y\right)$, whose volatility is $\sigma_{X/Y}$. (d) By (14) with $N = Y$, $V = Y_0 \mathbb{E}^Y\left[(X_T/Y_T - 1)^+\right]$, which is Y_0 times the Black-Scholes price of a call on an asset worth X_0/Y_0, with strike 1, zero rate and volatility $\sigma_{X/Y}$. (e) $\sigma_{X/Y} = 0.15\sqrt{2(1-\rho)}$ and $V = 2N(\sigma_{X/Y}/2) - 1$: 0.119 for $\rho = -1$, 0.084 for $\rho = 0$, 0.060 for $\rho = 0.5$, 0.027 for $\rho = 0.9$ and 0 for $\rho = 1$. The option is an option on the relative performance of the two indices; the more correlated they are, the less this relative performance moves.

Solution 3.3

(a) S^f is a foreign traded asset; converted into domestic currency at any time, XS^f is a domestic traded asset. Proposition 6 with $N^1 = B$ and $N^2 = XB^f$ gives (20). (b) The drift of Z is $\mu_X + r^f - r$ and Z is a martingale. Then $d(1/z) = -dz/z^2 + d\langle z\rangle/z^3$ with $dZ = Z\sigma_X\, dW^X$ gives the result (the dt term is $+\sigma_X^2$). (c) $d\left\langle W^f, \ln Z^{-1}\right\rangle = -\rho\sigma_X\, dt$, so $W^f + \rho\sigma_X t$ is a $\mathbb{Q}$-Brownian motion by Theorem 2. Substituting $dW^f = dW^S - \rho\sigma_X\, dt$ in the $\mathbb{Q}^f$-dynamics gives (21). (d) By (22) with $T = 1$, the value is $e^{(r^f - r - \rho\sigma_N\sigma_X)} = e^{-0.02\rho}$, between 1.0202 and 0.9802. A positive correlation between the Nasdaq and the dollar lowers the price.

Solution 3.4

(a) By (13) with $N^1 = P(\cdot, S)$ and $N^2 = P(\cdot, T)$, the density on $\mathcal{F}_T$ is $\frac{P(0,S)}{P(0,T)}\frac{P(T,T)}{P(T,S)} = \frac{1+\tau L(T,S)}{1+\tau F(0;T,S)}$. (b) $\mathbb{E}^T[L] = \frac{\mathbb{E}^S[L(1+\tau L)]}{1+\tau F(0)} = \frac{F(0)+\tau\mathbb{E}^S[L^2]}{1+\tau F(0)}$ since $\mathbb{E}^S[L] = F(0)$, and $\mathbb{E}^S[L^2] = \mathrm{Var}^S(L) + F(0)^2$ gives the result. Since $\mathbb{E}^T[F(T;T,S)] > F(0;T,S)$ as soon as L is random, F is not a $\mathbb{Q}^T$-martingale. This is the convexity adjustment of Section 5.

Solution 4.1

(a) $r(t) = \theta + e^{-kt}\left(r_0 - \theta + \sigma\int_0^t e^{ku}dW_u\right)$. By the product rule,

$$dr = -ke^{-kt}\left(r_0 - \theta + \sigma\int_0^t e^{ku}dW_u\right)dt + \sigma\, dW_t = k(\theta - r)\, dt + \sigma\, dW_t.$$

(b) $\mathbb{E}[r(t)] = r_0e^{-kt} + \theta(1 - e^{-kt})$ and, by Itô's isometry, $\mathrm{Var}(r(t)) = \sigma^2\int_0^t e^{-2k(t-u)}du = \frac{\sigma^2}{2k}\left(1 - e^{-2kt}\right)$. (c) The time scale is $1/k$. For $kt \ll 1$, $\mathrm{Var}(r(t)) \approx \sigma^2 t$ and $\mathbb{E}[r(t)] \approx r_0$, as for $r_0 + \sigma W_t$ ($k = 0$). For $kt \gg 1$, the law of $r(t)$ tends to $\mathcal{N}(\theta, \sigma^2/(2k))$ whereas the variance of the Brownian motion grows without bound. (d) With $A(t) = -k$, $a(t) = k\theta$, $\sigma_1 = 0$ and $\sigma_2 = \sigma$, $Z_t = e^{-kt}$ and $r(t) = e^{-kt}\left[r_0 + \int_0^t k\theta e^{ku}du + \int_0^t \sigma e^{ku}dW_u\right]$, which is the formula of (a).

Solution 4.2

(a) See the proof of Proposition 9, and integrate (25) between t and T: $\int_t^T e^{-k(s-t)}ds = A(t,T)$. (b) $\xi(0,T) - \mathbb{E}[\xi(0,T)] = \int_0^T (r(u) - \mathbb{E}[r(u)])\, du$; the variance of this integral is the double integral of the covariance (Fubini). The computation is done in Route 1 above. (c) Proposition 10. (d) When $\tau = T - t \to 0$, $A \sim \tau$ and $C = O(\tau^2)$, so $R(t,T) \to r(t)$. When $\tau \to \infty$, $A \to 1/k$ and $R(t,T) \to R_\infty = \theta - \sigma^2/(2k^2)$, which does not depend on $r(t)$.

Solution 4.3

(a) $R(t,T) = \bar{A}(t,T)r(t) + C(t,T)/(T-t)$, so the Brownian part of $dR(t,T)$ is $\bar{A}(t,T)\sigma\, dW_t$. (b) The correlation is $\frac{\sigma^2\bar{A}_1\bar{A}_2}{\sqrt{\sigma^2\bar{A}_1^2\sigma^2\bar{A}_2^2}} = 1$. All rates move in the same direction, with amplitudes decreasing with the maturity: the model cannot produce changes of slope driven by a separate source of risk.

Solution 4.4

In one year, the mean is $3\% - 2\% \times e^{-0.3} = 1.518\%$ and the standard deviation $1\% \times \sqrt{(1 - e^{-0.6})/0.6} = 0.867\%$, hence $N(-1.518/0.867) = N(-1.75) = 4.0\%$. In the long run, the law is $\mathcal{N}(3\%, (1.291\%)^2)$ and the probability is $N(-2.32) = 1.0\%$.

Solution 4.5

(a) By (30), $P(0,1) = 0.970459$ and $P(0,1.5) = 0.956036$, hence $F(0;1,1.5) = 3.017\%$. (b) $\sigma_p = 1\% \times A(1,1.5) \times \sqrt{(1 - e^{-0.6})/0.6} = 0.403\%$ and the caplet is $1.015 \times ZBP(0,1,1.5,1/1.015) = 0.001600$, a value confirmed by a Monte Carlo simulation. (c) Solving $0.5 \times P(0,1.5)\,\mathrm{Bl}(F,K,v) = 0.001600$ gives $v = 27.3\%$: the Vasicek model produces a Black volatility which depends on the strike and the maturity.

Solution 4.6

(a) As in the Vasicek case with $\theta = 0$. By Itô's isometry, $\mathrm{Cov}(x_i(t), x_j(t)) = \rho_{ij}\sigma_i\sigma_j \int_0^t e^{-(a_i+a_j)(t-u)}du = \frac{\rho_{ij}\sigma_i\sigma_j}{a_i+a_j}\left(1 - e^{-(a_i+a_j)t}\right)$, and $\mathrm{Var}(r(t)) = \sum_{i,j}\mathrm{Cov}(x_i(t), x_j(t))$. (b) See the proofs in the text; the bond price uses the Laplace transform of the Gaussian variable $\sum_i I_i$. (c) With $\tau = T - t$, $M = \tau - \frac{a\tau^2}{2} + \frac{a^2\tau^3}{6} + O(a^3)$ and $M_{11} = \frac{1-e^{-2a\tau}}{2a} = \tau - a\tau^2 + \frac{2a^2\tau^3}{3} + O(a^3)$, so $\tau - 2M + M_{11} = \frac{a^2\tau^3}{3} + O(a^3)$ and $V = \frac{\sigma^2}{a^2}(\tau - 2M + M_{11}) \to \frac{\sigma^2\tau^3}{3}$. Finally $\frac{\sigma^2}{2a^2}\left(1 - e^{-aT}\right)^2 \to \frac{\sigma^2 T^2}{2}$.

Solution 4.7

(a) With $\tau = T - t$, $\partial_t p = (A'r + C')p$, $\partial_r p = -Ap$ and $\partial_{rr} p = A^2 p$. Dividing the PDE by p,

$$r\left(A' + kA + \frac{1}{2}\sigma^2 A^2 - 1\right) + (C' - k\theta A) = 0$$

for all r, hence the two equations. (b) Direct differentiation (or a numerical check). When $\sigma \to 0$, $h \to k$ and $A \to \frac{2(e^{k\tau}-1)}{2ke^{k\tau}} = \frac{1-e^{-k\tau}}{k}$. (c) $2k\theta = 0.03 \geq \sigma^2 = 0.01$: yes, the short rate stays positive.

Solution 5.1

(a) $v = 0.2$, $d_{1,2} = \pm 0.1$ and $\mathrm{Bl} = 0.03\,(N(0.1) - N(-0.1)) = 0.0023897$, so the caplet is worth $10^7 \times 0.5 \times 0.955 \times 0.0023897 = 11{,}411$. (b) Caplet minus floorlet is the FRA, worth $N\tau P(0,S)(F-K) = 0$ at the money: the floorlet is also worth 11,411. (c) The rate is random until its fixing date T only; the payment at S is taken into account by the discount factor $P(0,S)$, and τ multiplies the payoff.

Solution 5.2

(a) $F(T) = F(t) + \sigma_N\sqrt{T-t}\,G$ with $G \sim \mathcal{N}(0,1)$. With $s = \sigma_N\sqrt{T-t}$, $\mathbb{E}\left[(F(t) - K + sG)^+\right] = \int_{-d}^{\infty}(F(t) - K + sg)n(g)\,dg = (F(t)-K)N(d) + s\,n(d)$, since $\int_{-d}^{\infty} g\,n(g)\,dg = n(d)$. (b) At the money, $d = 0$ and $n(0) = 1/\sqrt{2\pi}$. In Black's formula at the money, $F\,(N(v/2) - N(-v/2)) \approx Fv\,n(0)$ for small v. Hence $\sigma_N \approx F\bar{\sigma}$: a 20% lognormal volatility on a 3% rate corresponds to a normal volatility of 60 basis points per year.

Solution 5.3

(a) Proposition 15 with $k = 1 < i = 2$: $dF_1 = \sigma_1 F_1\left(-\frac{\rho_{12}\tau_2\sigma_2 F_2}{1+\tau_2 F_2}dt + dZ_1^2\right)$. (b) The frozen drift of dF_1/F_1 is $-0.2 \times 0.8 \times 0.2 \times 0.03/1.03 = -0.093\%$ per year, hence $\mathbb{E}^2\left[F_1(1)\right] \approx 3\% \times e^{-0.00093} = 2.9972\%$: the adjustment is only -0.28 basis point, which explains why the frozen drift approximation works well.

Solution 5.4

(a) The in-arrears payment is worth $\tau P(t,T)\mathbb{E}^T[L(T,S) \mid \mathcal{F}_t]$ and, from the text, $\tau P(t,S)F(t)\left(1 + \tau F(t)e^{v^2}\right)$. Since $P(t,S)/P(t,T) = 1/(1+\tau F(t))$, $\mathbb{E}^T[L] = \frac{F+\tau F^2 e^{v^2}}{1+\tau F} = F + \frac{\tau F^2(e^{v^2}-1)}{1+\tau F}$. In Exercise 3.4, $\mathrm{Var}^S(L) = F^2(e^{v^2} - 1)$ for a lognormal L, which gives the same result. (b) $0.03^2(e^{0.4} - 1)/1.03 = 4.3$ basis points, and $0.03^2(e^{0.45} - 1)/1.03 = 5.0$ basis points.

Solution 6.1

(a) See the proof in the text. (b) For $t \leq T_\alpha$, the floating payment at T_k is worth $N\Delta_k P(t,T_k)R_k(t) = N\left(P(t,T_{k-1}) - P(t,T_k)\right)$, so that the receiver swap is worth $N\left[KC_{\alpha,\beta}(t) - (P(t,T_\alpha) - P(t,T_\beta))\right]$, as in (3).

Solution 6.2

Daily: $(1 + 0.02/360)^{91} - 1 = 0.506822\%$. Continuous: $e^{0.02\times 91/360} - 1 = 0.506836\%$. Simple: $0.02 \times 91/360 = 0.505556\%$. The difference between daily and continuous compounding is negligible (about 0.01 basis point), which justifies (44).

Solution 6.3

(a) On $[T_{k-1}, T_k]$, $\int_{T_{k-1}}^{T_k}\left(\frac{T_k - s}{\Delta_k}\right)^2 ds = \frac{\Delta_k}{3}$, and more generally $\int_t^{T_k}\left(\frac{T_k-s}{\Delta_k}\right)^2 ds = \frac{(T_k-t)^3}{3\Delta_k^2}$. (b) Forward-looking: $v = 0.3$ and $\mathrm{Bl} = 0.0023847$. Compounded RFR: $v = 0.3\sqrt{1 + 0.25/3} = 0.3122$ and $\mathrm{Bl} = 0.0024813$, about 4% more.

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