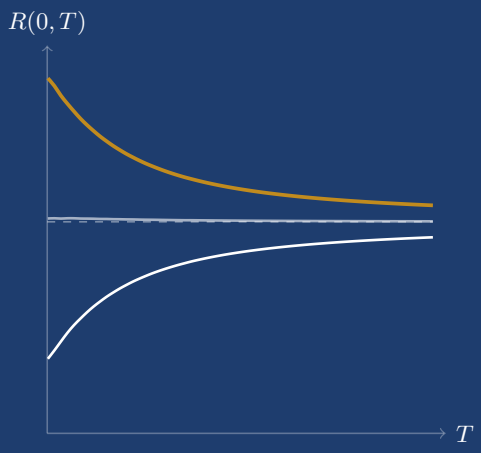
UNIVERSITÉ PARIS 1
PANTHÉON SORBONNE

Interest Rate Models

Exercise sessions with solutions

TD Book



Master MMMEF & IRFA
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Academic Year 2026–2027

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How to use this book. This book gathers the exercise sessions (TD) of the course *Interest Rate Models*: each chapter is one session, and the solutions are collected at the end of the book. References such as [Notes, Section 4] point to the lecture notes of the course, where the corresponding results are proved and discussed. A stochastic calculus cheatsheet is given in Appendix A.

Chapter 1

TD1: the Ornstein-Uhlenbeck process and the Vasicek model

Related reading: [Notes, Sections 4.1 and 4.2].

Let r be the solution of the stochastic differential equation

$$dr(t) = k\,(\theta - r(t))\,dt + \sigma\,dW_t, \qquad r(0) = r_0, \tag{1.1}$$

where W is a standard Brownian motion and k, θ, σ are positive constants. Throughout the chapter, $A(k,u) \coloneqq \frac{1}{k}\left(1 - e^{-ku}\right)$.

1.1 The Ornstein-Uhlenbeck process

Exercise 1.1 *Exponentially weighted moving average of a Brownian motion*

Let $I(t,T) \coloneqq \int_t^T e^{-k(T-s)} dW_s$ for $t \leq T$.

(a) Show that $\mathbb{E}\left(I(t,T) \mid \mathcal{F}_t\right) = 0$ and $\mathrm{Var}\left(I(t,T) \mid \mathcal{F}_t\right) = A(2k, T-t)$.

(b) Study the asymptotic behaviour of this variance when $T \to t$ and when $T \to +\infty$.

Exercise 1.2 *Solving the SDE*

For $t \leq T$, define $r(T) = \theta + (r(t) - \theta)\, e^{-k(T-t)} + \sigma I(t,T)$.

(a) Using Itô's lemma, check that r solves (1.1).

(b) Using the change of variables $Y(t) = e^{kt}\,(r(t) - \theta)$, show that (1.1) can be integrated and that its solution is given by the above formula.

(c) Show that $\mathbb{E}\left(r(T) \mid r(t)\right) = \theta + (r(t) - \theta)\, e^{-k(T-t)}$ and $\mathrm{Var}\left(r(T) \mid r(t)\right) = \sigma^2 A(2k, T-t)$.

(d) Show that, for $s, u \geq t$,

$$\gamma(t,s,u) \coloneqq \mathrm{Cov}\left(r(s), r(u) \mid r(t)\right) = \sigma^2 \mathbb{E}\left(I(t,s) I(t,u)\right) = \frac{\sigma^2}{2k} e^{-k(s+u)} \left(e^{2k(s \wedge u)} - e^{2kt}\right).$$

1.2 The integrated short rate

Exercise 1.3 *Mean and variance of the integrated short rate*

Define $\xi(t,T) = \int_t^T r(s)\, ds$.

(a) Show that $\mathbb{E}\left(\xi(t,T) \mid r(t)\right) = \theta(T-t) + (r(t) - \theta)\, A(k, T-t)$.

(b) Show that $\mathrm{Var}\left(\xi(t,T) \mid r(t)\right) = \int_t^T \int_t^T \gamma(t,s,u)\, ds\, du$.

(c) Show that $\mathrm{Var}\left(\xi(t,T) \mid r(t)\right) = \frac{\sigma^2}{k^2}\int_t^T \left[1 - 2e^{-k(u-t)} + e^{-2k(u-t)}\right] du$.

(d) Deduce that

$$V(t,T) \coloneqq \mathrm{Var}\left(\xi(t,T) \mid r(t)\right) = \frac{\sigma^2}{2k^3}\left[2k(T-t) - 3 + 4e^{-k(T-t)} - e^{-2k(T-t)}\right].$$

1.3 Zero-coupon rates

Exercise 1.4 *Zero-coupon rate and covariance structure*

The zero-coupon price and the zero-coupon rate are

$$P(t,T) = \mathbb{E}\left(e^{-\xi(t,T)} \mid \mathcal{F}_t\right), \qquad R(t,T) \coloneqq -\frac{1}{T-t}\ln P(t,T).$$

(a) Show that $R(t,T) = \bar{A}(k,T-t)r(t) + B(t,T)$ with

$$\bar{A}(k,T-t) = \frac{1}{T-t}A(k,T-t),$$
$$(T-t)B(t,T) = \left(\theta - \frac{\sigma^2}{2k^2}\right)\left((T-t) - A(k,T-t)\right) + \frac{\sigma^2}{4k}A(k,T-t)^2.$$

(b) Show that

$$d\left\langle R(\cdot,T_1), R(\cdot,T_2)\right\rangle_t = \sigma^2 \bar{A}(k,T_1-t)\bar{A}(k,T_2-t)\,dt.$$

(c) Deduce that the instantaneous correlation between $dR(t,T_1)$ and $dR(t,T_2)$ is equal to 1. Comment on the covariance structure of the yield curve in a one-factor Vasicek model.

Chapter 2

TD2: expectation calculations with change of measure

Related reading: [Notes, Section 3]. The cheatsheet of Appendix A is useful throughout this session.

In this chapter, $N(x) = \mathbb{E}\left(\mathbf{1}_{G \leq x}\right)$ where $G \sim \mathcal{N}(0,1)$, and interest rates are constant.

2.1 The Black-Scholes formula

Exercise 2.1 *Black-Scholes formula by change of measure*

Let $dS_t = S_t\left(r\,dt + \sigma\,dW_t\right)$ where W is a $\mathbb{Q}$-standard Brownian motion. We want to compute

$$\mathbb{E}\left[(S_T - K)^+\right] = \underbrace{\mathbb{E}\left[S_T \mathbf{1}_{S_T \geq K}\right]}_{B} - \underbrace{\mathbb{E}\left[K \mathbf{1}_{S_T \geq K}\right]}_{A}.$$

(a) Show that $S_t = S_0 \exp\left((r - \frac{\sigma^2}{2})t + \sigma W_t\right) \overset{law}{=} S_0 \exp\left((r - \frac{\sigma^2}{2})t + \sigma\sqrt{t}G\right)$.

(b) Show that $A = KN(d_A)$ with $d_A = \frac{\ln(S_0/K) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$.

(c) Let $Z_T \coloneqq \frac{S_T e^{-rT}}{S_0}$.

(i) Show that $\mathbb{Q}^S$ defined by $\frac{d\mathbb{Q}^S}{d\mathbb{Q}} = Z_T$ is a probability equivalent to $\mathbb{Q}$.

(ii) Show that $dZ_t = Z_t \sigma\,dW_t$.

(iii) Show that $dS_t = S_t\left((r + \sigma^2)dt + \sigma\,dW_t^S\right)$ where W^S is a $\mathbb{Q}^S$-standard Brownian motion.

(d) Show that $B = S_0 e^{rT} \mathbb{Q}^S(S_T \geq K) = S_0 e^{rT} N(d_B)$ with $d_B = \frac{\ln(S_0/K) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$.

(e) Deduce the Black-Scholes formula for the price $e^{-rT}\mathbb{E}\left[(S_T - K)^+\right]$ of the call.

2.2 The Margrabe formula

Exercise 2.2 *Exchange option*

Let

$$\begin{aligned} dX_t &= X_t\left(r\,dt + \sigma_X\,dW_t^X\right), \\ dY_t &= Y_t\left(r\,dt + \sigma_Y\,dW_t^Y\right), \end{aligned}$$

with $d\left\langle W^X, W^Y\right\rangle_t = \rho\,dt$. We want to compute the value of the exchange option $V = \mathbb{E}\left[(X_T - Y_T)^+\right]$, rewritten as $V = \mathbb{E}\left[Y_T\left(\frac{X_T}{Y_T} - 1\right)^+\right]$.

(a) Show that $d\left(\frac{1}{Y_t}\right) = \frac{1}{Y_t}\left(-(r-\sigma_Y^2)\,dt - \sigma_Y\,dW_t^Y\right)$.

(b) Show that $d\left(\frac{X_t}{Y_t}\right) = \frac{X_t}{Y_t}\left((\sigma_Y^2 - \rho\sigma_X\sigma_Y)\,dt + \sigma_X\,dW_t^X - \sigma_Y\,dW_t^Y\right)$.

(c) Define the density $Z_T := \frac{Y_T e^{-rT}}{Y_0}$ and P^2 the corresponding probability.

(i) Show that $dZ_t = Z_t\sigma_Y\,dW_t^Y$.

(ii) Show that $W_t^{2,X} = W_t^X - \rho\sigma_Y t$ and $W_t^{2,Y} = W_t^Y - \sigma_Y t$ are P^2-standard Brownian motions.

(iii) Show that $d\left(\frac{X_t}{Y_t}\right) = \frac{X_t}{Y_t}\left(\sigma_X\,dW_t^{2,X} - \sigma_Y\,dW_t^{2,Y}\right) = \frac{X_t}{Y_t}\sigma_{X/Y}\,dW_t^2$ with $\sigma_{X/Y}^2 = \sigma_X^2 + \sigma_Y^2 - 2\rho\sigma_X\sigma_Y$ and W^2 a P^2-standard Brownian motion.

(d) Show that $V = Y_0 e^{rT}\mathbb{E}^{P^2}\left[(M-1)^+\right]$ where $M \overset{law}{=} \frac{X_0}{Y_0}\exp\left(-\frac{1}{2}\sigma_{X/Y}^2 T + \sigma_{X/Y}\sqrt{T}G\right)$.

(e) Show that $e^{-rT}V$ is Y_0 times the Black-Scholes price of a call with spot X_0/Y_0, strike 1, volatility $\sigma_{X/Y}$ and zero interest rate.

(f) Application. Let X be the Nasdaq and Y the S&P 500, both with a volatility of 15%, $X_0 = Y_0 = 1$ and $T = 1$. Graph the price as a function of ρ and comment.

2.3 Foreign assets and quanto adjustment

Exercise 2.3 *Quanto adjustment*

Let S^f be a foreign asset denominated in a foreign currency, and X_t^f the value of one unit of foreign currency in domestic currency. Quantities with exponent f refer to the foreign economy: B^f is the foreign bank account and B the domestic one. Recall that the bank account has no Brownian part.

(a) Explain why $\frac{S^f}{B^f}$ is a $\mathbb{Q}^f$-martingale and why $\frac{S^f X^f}{B}$ is a $\mathbb{Q}$-martingale.

(b) Show that $Z_t = \frac{X_t^f B_t^f}{X_0^f B_t}$ is the density process of $\mathbb{Q}^f$ with respect to $\mathbb{Q}$.

(c) Let $dX_t^f = X_t^f\left(\mu_X\,dt + \sigma_X\,dW_t\right)$, where W is a $\mathbb{Q}$-Brownian motion.

(i) Show that necessarily $\mu_X = r - r^f$.

(ii) Show that $d\left(Z_t^{-1}\right) = Z_t^{-1}\left(\sigma_X^2\,dt - \sigma_X\,dW_t\right)$.

(d) Let $dS_t^f = S_t^f\left(r^f dt + \sigma_S\,dW_t^f\right)$, where W^f is a $\mathbb{Q}^f$-Brownian motion with $d\left\langle W^f, W\right\rangle_t = \rho\,dt$.

(i) Show that $dW_t^2 = dW_t^f + \rho\sigma_X\,dt$ defines a $\mathbb{Q}$-standard Brownian motion.

(ii) Show that $dS_t^f = S_t^f\left((r^f - \rho\sigma_S\sigma_X)\,dt + \sigma_S\,dW_t^2\right)$.

(e) Application. Let N_t be the value of the (USD denominated) Nasdaq and $I_t = N_t/N_0$.

(i) What is the value today of receiving I_1 EUR in one year?

(ii) Compute it as a function of ρ for $r = r^f = 0$, $\sigma_N = 20\%$ and $\sigma_{USD/EUR} = 10\%$. Comment.

Chapter 3

TD3: rehearsal before exam

Related reading: [Notes, Sections 2 to 4].

3.1 Vasicek model

The short rate at date t is denoted $r(t)$ and is the solution of

$$dr(t) = k\left(\theta - r(t)\right)dt + \sigma\, dW(t), \qquad r(0) = r_0. \tag{3.1}$$

Exercise 3.1

Show that $r_t = r_0 e^{-kt} + \theta\left(1 - e^{-kt}\right) + \sigma \int_0^t e^{-k(t-u)} dW_u$ is a solution of (3.1).

Exercise 3.2

Prove that:

(a) $\mathbb{E}\left(r_t\right) = r_0 e^{-kt} + \theta\left(1 - e^{-kt}\right)$;

(b) $\mathrm{Var}\left(r_t\right) = \frac{\sigma^2}{2k}\left(1 - e^{-2kt}\right)$.

Exercise 3.3

Discuss the behaviour of $r(t)$ for "small" t and "large" t. Characterise what is small and large, and compare with the Brownian case ($k = 0$).
Define the integrated short rate $X(t) \coloneqq \int_0^t r(s)\, ds$.

Exercise 3.4

Explain why $X(t)$ is a Gaussian process. Show that

$$\mathbb{E}\left(X(t)\right) = \theta t - \left(\theta - r_0\right)\frac{1 - e^{-kt}}{k}.$$

Exercise 3.5

Note $\phi(u, v) \coloneqq \mathrm{Cov}\left(r_u, r_v\right)$ and define $y(t) \coloneqq \int_0^t e^{-k(t-u)} dW_u$. Show that $\phi(u, v) = \sigma^2 \mathbb{E}\left(y(u)y(v)\right)$, then that

$$\phi(u, v) = \frac{\sigma^2}{2k} e^{-k(u+v)}\left(e^{2k(u \wedge v)} - 1\right).$$

Exercise 3.6

We want to compute $\mathrm{Var}\,(X_t)$.

(a) Explain why $\mathrm{Var}\,(X_t) = \sigma^2\,\mathrm{Var}\left(\int_0^t y(s)\,ds\right)$.

(b) Show that $\mathrm{Var}\,(X_t) = \int_0^t \int_0^t \phi(u,v)\,du\,dv$.

(c) Prove that $\mathrm{Var}\,(X(t)) = \frac{\sigma^2}{2k^3}\left(2kt - 3 + 4e^{-kt} - e^{-2kt}\right)$.

Exercise 3.7

Explain why the price of the zero-coupon bond of maturity T is $P(T) = \mathbb{E}\left(\exp\left(-X(T)\right)\right)$, the expectation being taken under the risk-neutral probability. Thanks to the previous computations, compute this price as a function of k, θ, σ, r_0 and T.

3.2 Around the Black-Scholes formula

Consider a situation where there is one riskless rate r and where the dynamics of some underlying price under the risk-neutral probability $\mathbb{Q}$ is

$$dS_t = S_t\left(c\,dt + \sigma\,dW_t\right), \qquad S_0 = 1,$$

where W is a standard Brownian motion, σ the volatility and c the cost of carry of the underlying. Define the forward price of maturity T as $F_T \equiv \mathbb{E}^{\mathbb{Q}}(S_T)$ and Y a process similar to S but without trend:

$$dY_t = Y_t\sigma\,dW_t, \qquad Y_0 = 1.$$

We want to compute the price of a call option of strike K and maturity T on S_T:

$$C = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[(S_T - K)^+\right] = e^{-rT}\left(B - A\right), \qquad B \equiv \mathbb{E}^{\mathbb{Q}}\left[S_T\mathbf{1}_{S_T \geq K}\right], \quad A \equiv \mathbb{E}^{\mathbb{Q}}\left[K\mathbf{1}_{S_T \geq K}\right].$$

Exercise 3.8

Discuss the value of c with respect to r when S is (a) a stock paying dividends, (b) an exchange rate, (c) a commodity contract.

Exercise 3.9

(a) Integrate S_T and Y_T.

(b) Compute F_T as a function of c and T.

(c) Show that $S_T = F_T Y_T$.

(d) Show that $A = KN(d_A)$ with $d_A = \frac{\ln(F_T/K) - \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}$.

(e) Show that $\mathbb{Q}^S$ defined by $\frac{d\mathbb{Q}^S}{d\mathbb{Q}} = Y_T$ is a probability equivalent to $\mathbb{Q}$.

(f) Show that $W_t^S \equiv W_t - \sigma t$ is a $\mathbb{Q}^S$-standard Brownian motion.

(g) Show that $dY_t = Y_t\left(\sigma^2 dt + \sigma\,dW_t^S\right)$.

(h) Show that $B = F_T\mathbb{Q}^S(S_T \geq K) = F_T N(d_B)$ with $d_B = \frac{\ln(F_T/K) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}$.

Exercise 3.10

(a) Thanks to the above, write C as a function of F_T, K, r, σ and T.

(b) Check that when $c = r$ the usual Black-Scholes formula is recovered.

3.3 Hull-White model

The short rate is modelled as $r(t) = \sum_{i=1}^{n} x_i(t) + \varphi(t)$, where

$$dx_i(t) = -a_i x_i(t)\,dt + \sigma_i\,dW_i(t), \qquad x_i(0) = 0,$$

with $d\langle W_i, W_j\rangle_t = \rho_{ij}\,dt$ ($\rho_{ii} = 1$), and φ is a deterministic function such that $\varphi(0) = r(0)$. In the Vasicek model, $n = 1$ and $\varphi(t) = r_0 e^{-kt} + \theta\left(1 - e^{-kt}\right)$.

Exercise 3.11

Demonstrate the following relationships:

$$\begin{aligned}
x_i(t) &= \sigma_i \int_0^t e^{-a_i(t-u)} dW_i(u),\\
\mathbb{E}\left(x_i(t) \mid \mathcal{F}_u\right) &= x_i(u) e^{-a_i(t-u)}, \quad u \le t,\\
\mathrm{Var}\left(x_i(t)\right) &= \frac{\sigma_i^2}{2a_i}\left(1 - e^{-2a_i t}\right),\\
\mathrm{Cov}\left(x_i(t), x_j(t)\right) &= \frac{\rho_{ij}\sigma_i\sigma_j}{a_i + a_j}\left(1 - e^{-(a_i+a_j)t}\right),\\
\gamma_{ij}(t,s,u) \coloneqq \mathrm{Cov}\left(x_i(s), x_j(u) \mid \mathcal{F}_t\right) &= \frac{\rho_{ij}\sigma_i\sigma_j}{a_i + a_j} e^{-a_i s} e^{-a_j u}\left(e^{(a_i+a_j)(s\wedge u)} - e^{(a_i+a_j)t}\right),\\
\mathrm{Var}\left(r(t)\right) &= \sum_{i=1}^{n} \mathrm{Var}\left(x_i(t)\right) + 2\sum_{i<j} \mathrm{Cov}\left(x_i(t), x_j(t)\right).
\end{aligned}$$

Exercise 3.12

Let $I_i(t,T) \equiv \int_t^T x_i(u)\,du$ and $M_i(t,T) = \frac{1-e^{-a_i(T-t)}}{a_i}$. Show that, conditionally on $\mathcal{F}_t$, the $I_i(t,T)$ are Gaussian with $\mathbb{E}\left(I_i(t,T) \mid \mathcal{F}_t\right) = M_i(t,T)x_i(t)$ and

$$\begin{aligned}
\mathrm{Var}\left(I_i(t,T) \mid \mathcal{F}_t\right) &= \frac{\sigma_i^2}{a_i^2}\left(T - t + \frac{2}{a_i}e^{-a_i(T-t)} - \frac{1}{2a_i}e^{-2a_i(T-t)} - \frac{3}{2a_i}\right),\\
\mathrm{Cov}\left(I_i(t,T), I_j(t,T) \mid \mathcal{F}_t\right) &= \frac{\rho_{ij}\sigma_i\sigma_j}{a_i a_j}\Big[(T-t) + \frac{e^{-a_i(T-t)} - 1}{a_i} + \frac{e^{-a_j(T-t)} - 1}{a_j}\\
&\qquad - \frac{e^{-(a_i+a_j)(T-t)} - 1}{a_i + a_j}\Big],
\end{aligned}$$

and deduce

$$V(t,T) \equiv \mathrm{Var}\left(\sum_i I_i(t,T) \mid \mathcal{F}_t\right) = \sum_i \mathrm{Var}\left(I_i(t,T) \mid \mathcal{F}_t\right) + 2\sum_{i<j} \mathrm{Cov}\left(I_i(t,T), I_j(t,T) \mid \mathcal{F}_t\right).$$

Exercise 3.13

(a) Show that the price of the zero-coupon bond is

$$P(t,T) = \exp\left\{-\phi(t,T) - \sum_{i=1}^{n} M_i(t,T)x_i(t) + \frac{1}{2}V(t,T)\right\}, \qquad \phi(t,T) \equiv \int_t^T \varphi(u)\,du.$$

(b) Show that ϕ is characterised by the following relationship between today's discount factors, and comment:

$$\exp\left(-\phi(t,T)\right) = \frac{P(0,T)}{P(0,t)} \exp\left(-\frac{1}{2}\left(V(0,T) - V(0,t)\right)\right).$$

(c) Prove that

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left(-\sum_{i=1}^{n} M_i(t,T)x_i(t) + \frac{1}{2}\Delta V(t,T)\right)$$

with $\Delta V(t,T) \equiv V(t,T) + V(0,t) - V(0,T)$.

3.4 Greeks, change of measure and a point mass at the strike

In the Black-Scholes model with interest rate r, write $S_T = S_0 X$ with $X = \exp\left((r - \frac{\sigma^2}{2})T + \sigma W_T\right)$ under $\mathbb{Q}$, and let $C(S_0) = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[(S_0 X - K)^+\right]$ be the price of a call. Let $Y = e^{-rT}X$ and p_X the density of X.

Exercise 3.14 *Delta and Gamma of a call*

(a) Show that the Delta of the call can be written as

$$\Delta(S_0) \coloneqq \frac{\partial C}{\partial S_0}(S_0) = \mathbb{E}^{\mathbb{Q}}\left[Y \mathbf{1}_{\{S_T > K\}}\right].$$

Justify carefully the differentiation under the expectation.

Let $(\delta_\varepsilon)_{\varepsilon>0}$ be the family of functions defined by $\delta_\varepsilon(u) = \frac{1}{2\varepsilon}$ if $|u| < \varepsilon$ and 0 otherwise.

(b) Show that $\int_{\mathbb{R}} \delta_\varepsilon(u)\, du = 1$.

(c) Show that, for any continuous function g, $\lim_{\varepsilon\to 0} \int_{\mathbb{R}} g(u)\delta_\varepsilon(u-a)\, du = g(a)$.

The formal limit of $\delta_\varepsilon(\cdot - a)$ is the Dirac mass δ_a, which satisfies $\int g(u)\delta_a(u)\, du = g(a)$.

(d) Admitting that the derivative, in the above limiting sense, of $u \mapsto \mathbf{1}_{\{u>0\}}$ is δ_0, show that $\frac{d}{dS_0}\mathbf{1}_{\{S_0 X > K\}} = X\delta_0(S_0 X - K)$.

(e) Deduce that the Gamma of the call satisfies

$$\Gamma(S_0) \coloneqq \frac{\partial^2 C}{\partial S_0^2}(S_0) = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[X^2 \delta_0(S_0 X - K)\right].$$

(f) Using the defining property of the Dirac mass and the density p_X, show that

$$\Gamma(S_0) = e^{-rT}\frac{K^2}{S_0^3}\, p_X\left(\frac{K}{S_0}\right).$$

(g) Check that this is the Black-Scholes Gamma $\frac{n(d_1)}{S_0\sigma\sqrt{T}}$, where n is the standard normal density.

(h) Explain briefly why this result shows that the Gamma of a call option is concentrated around the strike.

Appendix A

Stochastic calculus cheatsheet

Change of measure. Let P^1 be a probability and P^2 an equivalent one with density process $Z_t = \frac{dP^2}{dP^1}\Big|_{\mathcal{F}_t}$. Then M is a P^2-martingale if and only if ZM is a P^1-martingale.

Girsanov. Let W^1 be a P^1-standard Brownian motion and Z as above. Then $W_t^2 = W_t^1 - \int_0^t \frac{d\langle Z, W^1\rangle_s}{Z_s}$ is a P^2-standard Brownian motion. If $dZ_t = Z_t\lambda_t\, dW_t$ with $d\langle W, W^1\rangle_t = \rho_t\, dt$, then $\frac{d\langle Z, W^1\rangle_t}{Z_t} = \rho_t\lambda_t\, dt$.

Product rule. For two diffusions X and Y, $d(X_tY_t) = X_t\, dY_t + Y_t\, dX_t + d\langle X, Y\rangle_t$.

Itô, one dimension. $df(t, X_t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}d\langle X\rangle_t$.

Itô, two dimensions.

$$df(X_t, Y_t) = \frac{\partial f}{\partial x}dX_t + \frac{\partial f}{\partial y}dY_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}d\langle X\rangle_t + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}d\langle Y\rangle_t + \frac{\partial^2 f}{\partial x\partial y}d\langle X, Y\rangle_t.$$

Itô, n dimensions. Let W be an n-dimensional standard Brownian motion and $dX_t = \mu\, dt + \sigma\, dW_t$, with $\mu \in \mathbb{R}^n$ and σ an $n \times n$ matrix. For $f : \mathbb{R}^n \to \mathbb{R}$ with gradient ∇f and Hessian H_f,

$$df = \nabla f^\top dX_t + \frac{1}{2}dX_t^\top H_f\, dX_t = \left(\nabla f^\top \mu + \frac{1}{2}\mathrm{tr}\left(\sigma^\top H_f \sigma\right)\right)dt + \nabla f^\top \sigma\, dW_t.$$

Itô's isometry. For deterministic (or adapted, square integrable) g, h and Brownian motions with $d\langle W, W'\rangle_t = \rho\, dt$, $\mathbb{E}\left[\int_0^t g(s)\, dW_s \int_0^t h(s)\, dW_s'\right] = \rho\int_0^t g(s)h(s)\, ds$.

Gaussian Laplace transform. If $G \sim \mathcal{N}(\mu, s^2)$, $\mathbb{E}\left(e^{\lambda G}\right) = e^{\lambda\mu + \frac{1}{2}\lambda^2 s^2}$.

Solutions

TD1: the Ornstein-Uhlenbeck process and the Vasicek model

Solution 1.1

(a) $I(t,T)$ is the stochastic integral of a deterministic function on $[t,T]$: it is independent of $\mathcal{F}_t$ and Gaussian with mean zero. By Itô's isometry, $\mathrm{Var}\left(I(t,T)\right) = \int_t^T e^{-2k(T-s)}ds = \frac{1-e^{-2k(T-t)}}{2k} = A(2k,T-t)$.

(b) When $T \to t$, $A(2k,T-t) \sim T-t$: the variance is that of the Brownian increment $W_T - W_t$. When $T \to +\infty$, it tends to $1/(2k)$: the exponential weighting forgets the old shocks and the variance remains bounded.

Solution 1.2

(a) As a function of T, $r(T) = \theta + e^{-kT}\left[(r(t)-\theta)\,e^{kt} + \sigma\int_t^T e^{ks}dW_s\right]$. By the product rule,

$$dr(T) = -ke^{-kT}\left[(r(t)-\theta)\,e^{kt} + \sigma\int_t^T e^{ks}dW_s\right]dT + \sigma\,dW_T = k\left(\theta - r(T)\right)dT + \sigma\,dW_T,$$

and the formula gives $r(t)$ for $T=t$.

(b) $dY(t) = e^{kt}\left(k\left(r(t)-\theta\right)dt + dr(t)\right) = \sigma e^{kt}dW_t$: the drift cancels. Hence $Y(T) = Y(t) + \sigma\int_t^T e^{ks}dW_s$, and multiplying by e^{-kT} gives the formula.

(c) The first two terms are known at t and $I(t,T)$ is independent of $\mathcal{F}_t$, with mean zero and variance $A(2k,T-t)$ by the previous exercise.

(d) Only the random terms contribute to the covariance. By Itô's isometry, $\mathbb{E}\left(I(t,s)I(t,u)\right) = \int_t^{s\wedge u} e^{-k(s-v)}e^{-k(u-v)}dv = e^{-k(s+u)}\frac{e^{2k(s\wedge u)}-e^{2kt}}{2k}$.

Solution 1.3

(a) Integrate the conditional mean of Exercise 1.2(c): $\int_t^T e^{-k(s-t)}ds = A(k,T-t)$.

(b) $\xi(t,T) - \mathbb{E}\left(\xi(t,T)\mid r(t)\right) = \sigma\int_t^T I(t,s)\,ds$. Its variance is $\mathbb{E}\left[\int_t^T\int_t^T \sigma^2 I(t,s)I(t,u)\,ds\,du\right] = \int_t^T\int_t^T \gamma(t,s,u)\,ds\,du$ by Fubini's theorem.

(c) The integrand is symmetric in (s,u) and $s\wedge u = s$ on $\{s<u\}$, so the integral is twice the integral over this half-square:

$$\begin{aligned} V &= \frac{\sigma^2}{k}\int_t^T e^{-ku}\left[\int_t^u\left(e^{ks} - e^{2kt}e^{-ks}\right)ds\right]du = \frac{\sigma^2}{k^2}\int_t^T e^{-ku}\left[\left(e^{ku}-e^{kt}\right) + e^{2kt}\left(e^{-ku}-e^{-kt}\right)\right]du \\ &= \frac{\sigma^2}{k^2}\int_t^T\left[1 - 2e^{-k(u-t)} + e^{-2k(u-t)}\right]du. \end{aligned}$$

(d) The three terms integrate to $T-t$, $-2A(k,T-t) = \frac{1}{2k}\left(4e^{-k(T-t)}-4\right)$ and $A(2k,T-t) = \frac{1}{2k}\left(1-e^{-2k(T-t)}\right)$, which gives the result.

Solution 1.4

(a) $\xi(t,T)$ is Gaussian conditionally on $\mathcal{F}_t$, so $P(t,T) = \exp\left(-\bar{\xi} + \frac{1}{2}V\right)$ with $\bar{\xi}$ and V from Exercise 1.3. Write $\tau = T - t$ and $A = A(k,\tau)$. Since $e^{-k\tau} = 1 - kA$, $2k\tau - 3 + 4e^{-k\tau} - e^{-2k\tau} = 2k(\tau - A) - k^2A^2$, hence $\frac{1}{2}V = \frac{\sigma^2}{2k^2}(\tau - A) - \frac{\sigma^2}{4k}A^2$ and

$$-\ln P(t,T) = \bar{\xi} - \frac{1}{2}V = Ar(t) + \left(\theta - \frac{\sigma^2}{2k^2}\right)(\tau - A) + \frac{\sigma^2}{4k}A^2.$$

Dividing by τ gives the result. When $\tau \to \infty$, R tends to $\theta - \sigma^2/(2k^2)$.

(b) $B(t,T)$ and $\bar{A}$ are deterministic, so the Brownian part of $dR(t,T)$ is $\bar{A}(k, T-t)\sigma\, dW_t$, and the cross-variation is the product of the two coefficients.

(c) The correlation is $\frac{\sigma^2\bar{A}_1\bar{A}_2}{\sqrt{\sigma^2\bar{A}_1^2\,\sigma^2\bar{A}_2^2}} = 1$. All rates are driven by the same Brownian motion: they move in the same direction, with amplitudes decreasing with the maturity ($\bar{A}$ decreases from 1 to 0). The curve cannot twist: changes of slope independent of the level require several factors [Notes, Section 4.8].

TD2: expectation calculations with change of measure

Solution 2.1

(a) By Itô's formula, $d\ln S_t = (r - \frac{\sigma^2}{2})dt + \sigma\, dW_t$, and $W_t \overset{law}{=} \sqrt{t}G$.

(b) $S_T \geq K \Leftrightarrow G \geq -d_A$, so $A = K\mathbb{Q}(G \geq -d_A) = KN(d_A)$ by symmetry of the normal law.

(c) (i) $Z_T > 0$ and $\mathbb{E}(Z_T) = \mathbb{E}\left(e^{\sigma W_T - \sigma^2 T/2}\right) = 1$. (ii) $Z_t = \mathbb{E}(Z_T \mid \mathcal{F}_t) = S_t e^{-rt}/S_0$ since discounted prices are $\mathbb{Q}$-martingales, and $d(S_t e^{-rt}) = S_t e^{-rt}\sigma\, dW_t$. (iii) By Girsanov's theorem, $W_t^S = W_t - \sigma t$ is a $\mathbb{Q}^S$-Brownian motion; substitute $dW_t = dW_t^S + \sigma\, dt$.

(d) $B = S_0 e^{rT}\mathbb{E}\left[Z_T \mathbf{1}_{S_T \geq K}\right] = S_0 e^{rT}\mathbb{Q}^S(S_T \geq K)$. Under $\mathbb{Q}^S$, $\ln S_T = \ln S_0 + (r + \frac{\sigma^2}{2})T + \sigma W_T^S$, hence $\mathbb{Q}^S(S_T \geq K) = N(d_B)$.

(e) $e^{-rT}(B - A) = S_0 N(d_B) - Ke^{-rT}N(d_A)$: this is the Black-Scholes formula with $d_1 = d_B$ and $d_2 = d_A = d_1 - \sigma\sqrt{T}$.

Solution 2.2

(a) $d(1/y) = -dy/y^2 + d\langle y\rangle/y^3$, with $d\langle Y\rangle = \sigma_Y^2 Y^2 dt$.

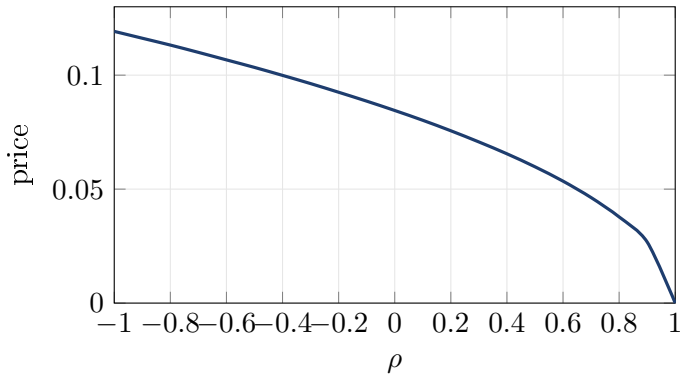
(b) By the product rule, $d(X/Y) = X\, d(1/Y) + \frac{1}{Y}dX + d\langle X, 1/Y\rangle$, and $d\langle X, 1/Y\rangle = -\frac{X}{Y}\rho\sigma_X\sigma_Y\, dt$; the terms in r cancel.

(c) (i) $Z_t = Y_t e^{-rt}/Y_0$ is a positive $\mathbb{Q}$-martingale with $Z_0 = 1$, and $dZ_t = Z_t\sigma_Y\, dW_t^Y$. (ii) By Girsanov's theorem, $W - \langle W, \ln Z\rangle$ is a P^2-Brownian motion, with $d\langle W^X, \ln Z\rangle = \rho\sigma_Y\, dt$ and $d\langle W^Y, \ln Z\rangle = \sigma_Y\, dt$. (iii) Substituting in (b), the drift becomes $\sigma_Y^2 - \rho\sigma_X\sigma_Y + \rho\sigma_X\sigma_Y - \sigma_Y^2 = 0$. The martingale $\sigma_X W^{2,X} - \sigma_Y W^{2,Y}$ has quadratic variation $\sigma_{X/Y}^2 t$; divided by $\sigma_{X/Y}$, it is a Brownian motion by Lévy's characterisation.

(d) $V = \mathbb{E}\left[Y_T(X_T/Y_T - 1)^+\right] = Y_0 e^{rT}\mathbb{E}\left[Z_T(X_T/Y_T - 1)^+\right] = Y_0 e^{rT}\mathbb{E}^{P^2}\left[(M-1)^+\right]$ with $M = X_T/Y_T$, which is lognormal under P^2 by (c).

(e) $\mathbb{E}^{P^2}\left[(M-1)^+\right]$ is the Black-Scholes price with zero rate of a call with spot X_0/Y_0 and strike 1, hence $e^{-rT}V = X_0 N(d_1) - Y_0 N(d_2)$ with $d_{1,2} = \frac{\ln(X_0/Y_0) \pm \frac{1}{2}\sigma_{X/Y}^2 T}{\sigma_{X/Y}\sqrt{T}}$.

(f) $\sigma_{X/Y} = 0.15\sqrt{2(1-\rho)}$ and the price is $2N(\sigma_{X/Y}/2) - 1$: 0.119 for $\rho = -1$, 0.084 for $\rho = 0$, 0.060 for $\rho = 0.5$, 0.027 for $\rho = 0.9$ and 0 for $\rho = 1$.



The option pays the relative outperformance of the Nasdaq over the S&P 500. The more correlated the two indices, the less their ratio moves, and the cheaper the option; it is worthless when $\rho = 1$, since the two indices then move together.

Solution 2.3

(a) S^f is a traded foreign asset: in units of the foreign bank account, its price is a martingale under the foreign risk-neutral measure. Since it can be converted into domestic currency at any time, $S^f X^f$ is the price of a domestic traded asset, hence $S^f X^f / B$ is a $\mathbb{Q}$-martingale.

(b) Z is positive with $Z_0 = 1$, and it is a $\mathbb{Q}$-martingale by (a) applied to $S^f = B^f$. For any foreign asset, $\frac{S^f}{B^f} Z = \frac{S^f X^f}{X_0^f B}$ is a $\mathbb{Q}$-martingale, so S^f / B^f is a $\mathbb{Q}^f$-martingale (a process M is a $\mathbb{Q}^f$-martingale if and only if MZ is a $\mathbb{Q}$-martingale). Hence Z is the density process of $\mathbb{Q}^f$.

(c) (i) By Itô's formula, $dZ_t = Z_t \left((\mu_X + r^f - r)\, dt + \sigma_X\, dW_t \right)$, and Z is a martingale: $\mu_X = r - r^f$. (ii) $d(1/z) = -dz/z^2 + d\langle z \rangle / z^3$ with $dZ = Z \sigma_X\, dW$ gives $d(Z^{-1}) = Z^{-1} \left(-\sigma_X\, dW + \sigma_X^2\, dt \right)$.

(d) (i) Z^{-1} is the density process of $\mathbb{Q}$ with respect to $\mathbb{Q}^f$. By Girsanov's theorem, $W^f - \left\langle W^f, \ln Z^{-1} \right\rangle$ is a $\mathbb{Q}$-Brownian motion, with $d\left\langle W^f, \ln Z^{-1} \right\rangle = -\rho \sigma_X\, dt$ (quadratic covariations do not depend on the measure). (ii) Substitute $dW^f = dW^2 - \rho \sigma_X\, dt$.

(e) (i) With $S^f = N$, the value is $e^{-r} \mathbb{E}^{\mathbb{Q}}(I_1) = e^{r^f - r - \rho \sigma_N \sigma_X}$. (ii) $e^{-0.02\rho}$: 1.0202 for $\rho = -1$, 1.0101 for $\rho = -0.5$, 1 for $\rho = 0$, 0.9900 for $\rho = 0.5$ and 0.9802 for $\rho = 1$. When the Nasdaq tends to rise together with the dollar ($\rho > 0$), its value in euros gains on both legs; since this value must grow at the euro rate under $\mathbb{Q}$, the Nasdaq performance itself grows more slowly, and the quanto payoff, which ignores the currency, is cheaper.

TD3: rehearsal before exam

Solution 3.1

This is Exercise 1.2(a) with $t = 0$: by the product rule applied to $r_t = \theta + e^{-kt} \left(r_0 - \theta + \sigma \int_0^t e^{ku} dW_u \right)$, $dr_t = k(\theta - r_t)\, dt + \sigma\, dW_t$.

Solution 3.2

(a) The stochastic integral has mean zero. (b) By Itô's isometry, $\mathrm{Var}(r_t) = \sigma^2 \int_0^t e^{-2k(t-u)} du = \frac{\sigma^2}{2k} \left(1 - e^{-2kt} \right)$.

Solution 3.3

The time scale is $1/k$. For $kt \ll 1$, $\mathbb{E}(r_t) \approx r_0$ and $\mathrm{Var}(r_t) \approx \sigma^2 t$: the short rate behaves like $r_0 + \sigma W_t$, which is exactly the case $k = 0$. For $kt \gg 1$, the law of r_t tends to $\mathcal{N}\left(\theta, \sigma^2/(2k)\right)$: the initial value is forgotten and the variance is bounded, whereas the variance of the Brownian motion grows without bound.

Solution 3.4

r is a Gaussian process (an affine function of a Wiener integral with a deterministic integrand) and $X(t)$ is a limit of linear combinations of its values, hence Gaussian. Integrating (a) of the second exercise of this section, $\mathbb{E}(X(t)) = \int_0^t \left(\theta + (r_0 - \theta)e^{-ks}\right) ds = \theta t - (\theta - r_0)\frac{1-e^{-kt}}{k}$.

Solution 3.5

$r_u - \mathbb{E}(r_u) = \sigma y(u)$, hence the first equality. By Itô's isometry,

$$\mathbb{E}\left(y(u)y(v)\right) = \int_0^{u\wedge v} e^{-k(u-s)}e^{-k(v-s)}ds = e^{-k(u+v)}\frac{e^{2k(u\wedge v)}-1}{2k}.$$

This is Exercise 1.2(d) with $t = 0$.

Solution 3.6

(a) $X_t - \mathbb{E}(X_t) = \sigma \int_0^t y(s)\, ds$. (b) The variance of an integral is the double integral of the covariance (Fubini's theorem). (c) See Exercise 1.3(c) and (d) with $t = 0$.

Solution 3.7

Under the risk-neutral probability, the price of a payoff is the expectation of its value discounted with the bank account, and the zero-coupon pays 1 at T. Since $X(T)$ is Gaussian, the price is $\exp\left(-\mathbb{E}(X(T)) + \frac{1}{2}\operatorname{Var}(X(T))\right)$, that is

$$P(T) = \exp\left(-\theta T + (\theta - r_0)\frac{1-e^{-kT}}{k} + \frac{\sigma^2}{4k^3}\left(2kT - 3 + 4e^{-kT} - e^{-2kT}\right)\right).$$

Solution 3.8

(a) With a continuous dividend yield q, the stock with reinvested dividends $S_t e^{qt}$ is traded and drifts at r: $c = r - q$. (b) Holding the foreign currency means investing it at the foreign rate: $c = r - r^f$. (c) Holding a commodity costs storage u and brings a convenience yield y: $c = r + u - y$.

Solution 3.9

(a) $S_T = \exp\left((c - \frac{\sigma^2}{2})T + \sigma W_T\right)$ and $Y_T = \exp\left(-\frac{\sigma^2}{2}T + \sigma W_T\right)$. (b) $F_T = e^{cT}$. (c) Immediate from (a) and (b). (d) $S_T \geq K \Leftrightarrow W_T/\sqrt{T} \geq -d_A$, hence $A = KN(d_A)$. (e) $Y_T > 0$ and $\mathbb{E}(Y_T) = 1$. (f) The density process is Y_t and $dY_t = Y_t\sigma\, dW_t$, so by Girsanov's theorem $W_t - \sigma t$ is a $\mathbb{Q}^S$-Brownian motion. (g) Substitute $dW_t = dW_t^S + \sigma\, dt$. (h) $B = F_T\mathbb{E}\left[Y_T \mathbf{1}_{S_T \geq K}\right] = F_T\mathbb{Q}^S(S_T \geq K)$, and under $\mathbb{Q}^S$, $\ln S_T = \ln F_T + \frac{\sigma^2}{2}T + \sigma W_T^S$.

Solution 3.10

(a) $C = e^{-rT}\left(F_T N(d_B) - KN(d_A)\right)$: this is Black's formula applied to the forward price. (b) When $c = r$ (and $S_0 = 1$), $e^{-rT}F_T = S_0$ and $C = S_0N(d_1) - Ke^{-rT}N(d_2)$ with $d_1 = d_B$ and $d_2 = d_A$.

Solution 3.11

Each x_i is an Ornstein-Uhlenbeck process with $\theta = 0$ and $x_i(0) = 0$: the first two relations follow from Exercise 1.2. By Itô's isometry with correlated Brownian motions, $\operatorname{Cov}\left(x_i(s), x_j(u) \mid \mathcal{F}_t\right) = \rho_{ij}\sigma_i\sigma_j \int_t^{s\wedge u} e^{-a_i(s-v)}e^{-a_j(u-v)}dv$, which gives γ_{ij}, and the two previous lines for $t = 0$ and $s = u$. The last line is the variance of a sum, φ being deterministic.

Solution 3.12

Integrating $x_i(u) = x_i(t)e^{-a_i(u-t)} + \sigma_i \int_t^u e^{-a_i(u-s)} dW_i(s)$ and exchanging the integrals (stochastic Fubini theorem),

$$I_i(t,T) = M_i(t,T)x_i(t) + \sigma_i \int_t^T M_i(s,T)\, dW_i(s).$$

Hence the conditional mean, and by Itô's isometry, with $\tau = T - t$,

$$\mathrm{Cov}\,(I_i, I_j \mid \mathcal{F}_t) = \rho_{ij}\sigma_i\sigma_j \int_t^T M_i(s,T)M_j(s,T)\, ds = \frac{\rho_{ij}\sigma_i\sigma_j}{a_i a_j} \int_0^\tau \left(1 - e^{-a_i v}\right)\left(1 - e^{-a_j v}\right) dv,$$

which gives the covariance. For $i = j$ ($\rho_{ii} = 1$), the bracket becomes $\tau - \frac{2}{a_i}\left(1 - e^{-a_i\tau}\right) + \frac{1}{2a_i}\left(1 - e^{-2a_i\tau}\right)$, which is the variance formula. The last formula is the variance of a sum.

Solution 3.13

(a) $\int_t^T r(u)\, du = \phi(t,T) + \sum_i I_i(t,T)$ is Gaussian conditionally on $\mathcal{F}_t$, with mean $\phi(t,T) + \sum_i M_i x_i(t)$ and variance $V(t,T)$; use $\mathbb{E}\left(e^{-G}\right) = e^{-\mathbb{E}(G) + \frac{1}{2}\mathrm{Var}(G)}$ for a Gaussian G.

(b) At $t = 0$, $x_i(0) = 0$ and $P(0,T) = \exp\left(-\phi(0,T) + \frac{1}{2}V(0,T)\right)$, and similarly for $P(0,t)$. Since $\phi(t,T) = \phi(0,T) - \phi(0,t)$, the ratio gives the relation. Whatever the shape of today's curve, the deterministic function φ can be chosen to reproduce it exactly: the model is calibrated to the initial curve by construction.

(c) Substitute (b) into (a).

Solution 3.14

(a) For each ω, $S_0 \mapsto (S_0X - K)^+$ is differentiable except at $S_0 = K/X$ (an event of probability zero for a given S_0), with derivative $X\mathbf{1}_{\{S_0X>K\}}$. Its difference quotients are bounded by X, which is integrable, so the dominated convergence theorem allows the differentiation under the expectation: $\Delta = e^{-rT}\mathbb{E}\left[X\mathbf{1}_{\{S_T>K\}}\right] = \mathbb{E}\left[Y\mathbf{1}_{\{S_T>K\}}\right]$.

(b) $\int \delta_\varepsilon = 2\varepsilon \times \frac{1}{2\varepsilon} = 1$.

(c) $\int g(u)\delta_\varepsilon(u-a)\, du = \frac{1}{2\varepsilon}\int_{a-\varepsilon}^{a+\varepsilon} g(u)\, du$ is the mean of g on $[a-\varepsilon, a+\varepsilon]$, which tends to $g(a)$ by continuity.

(d) Chain rule: the derivative of $S_0 \mapsto \mathbf{1}_{\{S_0X-K>0\}}$ is $\delta_0(S_0X - K) \times X$.

(e) Differentiate $\Delta = e^{-rT}\mathbb{E}\left[X\mathbf{1}_{\{S_0X>K\}}\right]$ with (d).

(f) With the change of variable $u = S_0x$,

$$\mathbb{E}\left[X^2\delta_0(S_0X - K)\right] = \int x^2\delta_0(S_0x - K)p_X(x)\, dx = \int \frac{u^2}{S_0^2}\delta_0(u-K)p_X\left(\frac{u}{S_0}\right)\frac{du}{S_0} = \frac{K^2}{S_0^3}p_X\left(\frac{K}{S_0}\right).$$

Note the factor K^2/S_0^3, which comes from the change of variable inside the Dirac mass.

(g) $\ln X \sim \mathcal{N}\left((r - \frac{\sigma^2}{2})T, \sigma^2T\right)$, so $p_X(x) = \frac{n(z)}{x\sigma\sqrt{T}}$ with $z = \frac{\ln x - (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$. At $x = K/S_0$, $z = -d_2$ and $\Gamma = e^{-rT}\frac{K\, n(d_2)}{S_0^2\sigma\sqrt{T}}$. Since $S_0n(d_1) = Ke^{-rT}n(d_2)$, $\Gamma = \frac{n(d_1)}{S_0\sigma\sqrt{T}}$.

(h) Γ is proportional to the density of S_T at the strike: the convexity of the payoff is concentrated at its kink $S_T = K$. The hedge ratio changes fast when the underlying is likely to end close to the strike; as the maturity approaches, the density concentrates and the Gamma of an at-the-money option explodes.