Advanced Topics in Financial Modelling, notes

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Abstract

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1 Multivariate portfolio analysis

1.1 The Markowitz analysis

1.1.1 Wealth equation

Consider a situation where, between date 0 and date 1, your initial wealth w has to be invested into:

- n risky assets with value S_t^i such that the random return $\tilde{\mu}^i \quad | \quad S_1^i = S_0^i \times (1+\tilde{\mu}^i)$ is described as a random variable with mean μ and covariance matrix Σ. μ_i is often referred to as the expected return of asset i,
- a non risky asset which return is known beforehand to be equal to R. R is also the cost of borrowing money which can be done without constraint.

Let α be the proportion of w invested into the risky assets. Let $\alpha_0 = 1 - \sum_i \alpha_i$ the amount invested or borrowed to the bank. The final (random) wealth yields

$$\tilde{W} = w \times (1 + R + \alpha' (\tilde{\mu} - R\mathbb{I}_n))$$

where w stands for the initial wealth. Let us define the excess returns as $\tilde{\pi_i} \equiv \tilde{\mu_i} - R$ and the excess return of the porfolio as $\tilde{\pi}(\alpha) \equiv \frac{\tilde{W}}{w} - (1+R)$. The following relationship holds:

$$\tilde{\pi}(\alpha) = \alpha' \tilde{\pi} \tag{1}$$

Equation 1 shows that the excess return of the portfolio is a linear function of the excess return of the risky asset. That linear relationship between excess return is a key structure to keep in mind. Relation between portfolio moments and risky asset moments follows:

- $\pi(\alpha) = \mathbb{E}(\alpha' \tilde{\pi}) = \alpha' \bar{\pi}$
- $Var(\tilde{\pi}(\alpha)) = \mathbb{E}(\alpha' \tilde{\pi} \tilde{\pi}' \alpha) = \alpha' \Sigma \alpha$

Definition 1 (Sharpe ratio)**.** The Sharpe ratio of an asset with average Excess Return π and volatility σ is defined as the ratio of the two: $\frac{\pi}{\sigma}$.

1.1.2 Optimal portfolio

The optimal portfolio is the solution of the same maximisation equation

$$u(w) = \sup_{\alpha \in \mathbb{R}^n} E(U(\tilde{W})) \tag{2}$$

For a CARA utility with Gaussian returns, equation 2 becomes:

$$u(w) = \sup_{\alpha \in \mathbb{R}^n} \alpha' \pi - \frac{\lambda}{2} \alpha' \Sigma \alpha \tag{3}$$

The optimisation program 3 has to be solved in $\mathbb{R}^n$. It can be done line by line as in Markowitz seminal paper but it makes the result difficult to read.

It is somewhat easier to differentiate directly in $\mathbb R^n$ even if this requires prior knowledge of some ad-hoc rules. See [8] for instance. One finds that:

$$\bar{\alpha} = \frac{1}{\lambda}\Sigma^{-1}\pi \tag{4}$$

Define an auxiliary function $\phi_y^\alpha(\epsilon) = f(\alpha + \epsilon y)$

$$\alpha \text{ optimal} \iff \left.\frac{d\phi_y^\alpha(\epsilon)}{d\epsilon}\right|_{\epsilon=0} = 0 \ \forall y \in \mathbb{R}^n$$

Practically:

$$\begin{aligned} f(\alpha+\epsilon y) &= (\alpha+\epsilon y)'\pi - \frac{\lambda}{2}(\alpha+\epsilon y)'\Sigma(\alpha+\epsilon y) \\ &= cte + (y'\pi - \lambda\alpha'\Sigma y)\,\epsilon - \frac{\lambda}{2}y'\Sigma y\epsilon^2 \\ \left.\frac{d\phi_y^\alpha(\epsilon)}{d\epsilon}\right|_{\epsilon=0} &= (\pi - \lambda\Sigma\alpha)'\,y \end{aligned}$$

Since the above has to be equal to zero whatever y, QED.

Remark 1. In the non correlated case, it is optimal to invest in **all** assets, not only in the best performing...The amount of risk shall be proportionnal to the expected Sharpe ratio

1.1.3 The two fund separation theorem

Theorem 1 (The two funds separation theorem)**.** *Let P_m the portfolio defined by*

$$\begin{aligned} \alpha_m &= \frac{1}{\lambda_m}\Sigma^{-1}\pi \\ \lambda_m &= 1'\Sigma^{-1}\pi \end{aligned}$$

Any optimal portfolio is a combination of P_m and the cash account:

- $e = \lambda_m/\lambda$ invested in P_m
- $(1 - \lambda_m/\lambda)$ invested or borrowed from the cash account

1.2 Case with no riskless asset

In the case when the investor is constrained to invest only in equities then the program yields

$$\begin{aligned} u(w) = &\sup_{\alpha\in\mathbb{R}^n} \alpha'\pi - \frac{\lambda}{2}\alpha'\Sigma\alpha \\ s.t. \quad &\alpha' 1 = 1 \end{aligned}$$

In that case the solution can be obtained as

$$\alpha(\lambda) = \left(1 - \frac{\lambda_m}{\lambda}\right) P_{min} + \frac{\lambda_m}{\lambda} P_m$$

where $P_{min} \equiv \frac{1}{1'\Sigma^{-1}1}\Sigma^{-1}1$ is the minimum-variance portfolio.

1.3 Analysis in terms of eigen portfolios

1.3.1 Generic structure

Since Σ is symmetric, there exists an orthonormal matrix E and a diagonal matrix Λ such that $\Sigma = E\Lambda E'$. E is the matrix of independant eigen portfolios. Each column is an eigenportfolioo which weights can be read row by row. The optimal portfolio can be rewritten

$$\alpha = \frac{1}{\gamma} E' \Lambda^{-1} E \pi$$

or equivalently

$$E\alpha = \frac{1}{\gamma} \Lambda^{-1} E \pi$$

If one express portfolios in terms of eigenvectors, then...

1.3.2 2D example

Consider two assets with same volatility and μ_1 and μ_2 and correlation ρ

$$\begin{aligned}
\Sigma^{-1} &= \frac{1}{1-\rho^2} \begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix} \\
\begin{pmatrix} \alpha_1 \\ \alpha_1 \end{pmatrix} &= \frac{1}{1-\rho^2} \begin{pmatrix} \pi_1 - \rho\pi_2 \\ \pi_2 - \rho\pi_1 \end{pmatrix} \\
&= \frac{\pi_1 + \pi_2}{2(1+\rho)} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{\pi_1 - \pi_2}{2(1-\rho)} \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\end{aligned}$$

Combination of long and long/short portfolio. The higher the correlation, the more attractive the Long/short. Time series vs cross section momentum.

1.4 Risk contribution

1.4.1 Markowitz revisited

Theorem 2. *Euler decomposition of homogeneous functions. Let* $f : \mathbb{R}^n \to \mathbb{R}$ *so that* $\forall \lambda > 0 : f(\lambda x) = \lambda f(x)$ *then, necessarily*

$$f(x) = \sum x_i \frac{\partial f}{\partial x_i} = x' \nabla x$$

Proof. By differentiating $f(\lambda x)$ one gets: $df(\lambda x) = \sum x_i \frac{\partial f}{\partial x_i} d\lambda$. Now differentiating $\lambda f(x)$ yields that $\mathrm{d}(\lambda f(x)) = f(x)\, d\lambda$. Since this is true for any possible x then both differential are equal. □

Definition 2 (Volatility exposure)**.** Let Let $\Sigma \equiv V\Gamma V$ where V is the diagonal matrix with volatility terms (in finance, volatility = standard deviation) : $V_{ii} = \sqrt{\Sigma_{ii}}$, $V_{ij} = 0$ if $i \neq j$. The volatility exposure is defined as

$$vex \equiv Vx$$

and the vector of sharpe ratios as

$$sharpe \equiv V^{-1}\mu$$

Theorem 3 (Risk contributions)**.** *The marginal risk contribution as a function of weight is defined by*

$$\nabla_x \sigma = \frac{1}{\sigma}\Sigma x \text{ with } \sigma(x) = x'\nabla_x \sigma$$

or as a function of volatility exposures

$$\nabla_{vex} \sigma = \frac{1}{\sigma}\Gamma vex \text{ with } \sigma(vex) = vex'\nabla_{vex} \sigma$$

The total, or absolute, risk contribution is defined as

$$x \otimes \nabla_x \sigma = vex \otimes \nabla_{vex} \sigma$$

Proof. Risk is defined by

$$\begin{aligned} \sigma^2(x) &= x'\Sigma x \\ \Rightarrow 2\sigma d\sigma &= (2\Sigma x)' dx \\ \Rightarrow d\sigma &= \frac{1}{\sigma}(\Sigma x)' dx \end{aligned}$$

By definition $d\sigma \equiv (\nabla_x \sigma)' dx$ hence $\nabla_x \sigma = \frac{1}{\sigma}\Sigma x$. Now, Markowitz equation (4) can rewritten as

$$\begin{aligned} x &= \frac{1}{\gamma}\Sigma^{-1}\mu \\ \Longleftrightarrow \Sigma x &= \frac{1}{\gamma}\mu \\ \Longleftrightarrow \nabla_{vex} \sigma &= \Gamma^{-1} sharpe \end{aligned}$$

In plain English, the above equation can be formulated: as the marginal risk contributions, in terms of volatility exposure, has to be proportional to the expected vector of Sharpe ratios. □

1.4.2 Risk budgeting strategies

Since $\Gamma^{-1}sharpe$ might be complex to obtain, some people prefer to define it exogenously. This amounts to think in terms of risk-budgeting instead of "expectations" so they would solve

$$x \otimes \nabla_x \sigma = \begin{pmatrix} Budget_1 \\ \\ \\ Budget_n \end{pmatrix}$$

1.5 Transaction costs

1.5.1 General

We study the following program:

$$x^* = \left\{ \ argmax \ \pi' x - \tfrac{\lambda}{2} x' \Gamma x - k' \left| x - x_0 \right| \right. \tag{5}$$

where $x = \begin{pmatrix} x_1 \\ ... \\ x_n \end{pmatrix}$ is a vector of risk budgets, x' its transpose, $|x| = \begin{pmatrix} |x_1| \\ ... \\ |x_n| \end{pmatrix}$, Γ a $n \times n$ correlation matrix, x_0 the initial weight vector, π the vector of expected Sharpe ratios and k the vector of rebalancing penalties. When there is no rebalancing penalties, one finds that

$$x^* = \frac{1}{\lambda} \Gamma^{-1} \pi \equiv opt\left(\pi\right)$$

As a linear transformation of π, $opt\left(\pi\right)$ is not sparse.

The objective function is concave but not differentiable. In that case,x^* is optimal iff $0 \in \partial L\left(x^*\right)$.

As a reminder, in dimension 1, a sub-derivative of a convex function f in x_0 is a scalar s (slope) such that

$$\forall x : f\left(x\right) \geq f\left(x_0\right) + s\left(x - x_0\right)$$

The set of all sub-derivatives is called the sub-differential of f at x_0. It is sometimes denoted ∂f. On has $\underset{x \to w^{+/-}}{lim} \frac{f(x) - f(w)}{x - w} \in \partial f$. For any convex function, x_0 is a minimum iff $0 \in \partial f\left(x_0\right)$. Typically, $|x|$is convex. $\partial f\left(0\right) = \left[-1, 1\right]$. Since $0 \in \left[-1, 1\right], |x|$ is minimum at zero.

In dimension n, one can define the sub-gradient of $f : E \to R$ as the vector s such that

$$\forall x : f\left(x\right) \geq f\left(x_0\right) + \left\langle s, x - x_0 \right\rangle$$

The set of all sub-gradients is called the sub-differential of f at x_0. Equivalently, For any convex function, x_0 is a minimum iff $0 \in \partial f\left(x_0\right)$.

For any coordinate i

$$\forall x_i \neq x_{0i} : \partial_i L(x) = \pi_i - \lambda (\Gamma x)_i - k_i 1_{x_i > x_{0i}} + k_i 1_{x_i < x_{0i}}$$

For 0 to be in the interior of $\partial L(x)$, x must satisfy

$$\forall x_i : -k_i < \pi_i - \lambda (\Gamma x)_i < k_i$$

The set of all optimal point is thus the interior of the simplex defined by the 2^n points $\{opt(\pi + p \otimes k) \mid p \in [-1,1]^n\}$.

In dimension 2, this set is defined by the polygon $opt(\pi_1 - k_1, \pi_2 - k_2)$,$opt(\pi_1 - k_1, \pi_2 + k_2)$, $opt(\pi_1 + k_1, \pi_2 + k_2)$, $opt(\pi_1 + k_1, \pi_2 - k_2)$

As shown in the following graphs, this creat a "no trade zone":

1.6 1D case (to be rewritten)

1.6.1 Poids optimaux

Let $x_M = \lambda \frac{\pi}{\sigma^2}$ the optimal solution without transaction cost. The Lagrangian, L, is differentiable except at $x = x_0$. One has

$$\forall x \neq w : L'(x) = \pi - \frac{x\sigma^2}{\lambda} - k(1_{x>x_0} - 1_{x<x_0}) \tag{6}$$

If $x > x_0$, L' can be neagtive at $x_b = x_M - \frac{\lambda k}{\sigma^2}$. if $x < x_0$, it can get null at $x_s = x_M + \frac{\lambda k}{\sigma^2}$.

Case 1 : $x_0 \notin [x_b, x_s]$, $\quad x_b > x_0$ alors la stratégie optimale consiste à détenir x_b c'est à dire à acheter de l'actif. Si $x_s < x_0$, alors la stratégie optimale consiste à détenir x_s soit donc vendre de l'actif.

Case 2 : $x_0 \in [x_b, x_s]$ Dans ce cas, on voit que L' est positif sur $[x_b, w[$ et négatif sur $[w, x_s[$. La fonction valeur est donc maximale en x_0 sur l'intervalle ce qui entraîne $x^* = w$. Le segment $[x_B, x_S]$ définit donc une zone de non trading.

Case 3 : $x_b, x_s \notin [-b\lambda/\sigma, +b\lambda/\sigma], w \notin [x_b, x_s]$ Dans ce cas, on se ramène en $-b$ ou b selon le cas.

Case 4 : $x_b, x_s \notin [-b\lambda/\sigma, +b\lambda/\sigma], w \in [x_b, x_s]$ Dans ce cas, on se ramène aussi en $-b\lambda/\sigma$ ou $b\lambda/\sigma$ alors qu'on est censé être dans la zone de non trade mais là, on n'a pas le choix car sinon on nrespecte pas les contraintes

1.7 Optimisation "in practice"

`https://osqp.org/` Numerical solver to solve with additional constraints

Problem 5 can be rewritten as a quadratic program of higher dimension introducing $t = |x - x_0|$ and appropriate constraints:

$$\min_{x,t^+,t^-} \frac{\lambda}{2} x'\Gamma x - \pi' x + k' t$$
$$s.t. \begin{cases} x - t \leq x_0 \\ x_0 \leq x + t \end{cases}$$

Denoting $X = \begin{pmatrix} x \\ t \end{pmatrix}$, $P = \begin{pmatrix} \lambda\Gamma & 0 \\ 0 & 0 \end{pmatrix}$,$q = \begin{pmatrix} -\pi \\ k \end{pmatrix}$, $A = \begin{pmatrix} I_n & -I_n \\ I_n & I_n \end{pmatrix}$,$l = \begin{pmatrix} -\infty \\ x_0 \end{pmatrix}$, $u = \begin{pmatrix} x_0 \\ +\infty \end{pmatrix}$ and I_n the identity matrix of dimension n, problem 5 writes

$$\min_X \ \frac{1}{2} X' P X + q' X$$
$$s;t. \ l \leq AX \leq u$$

which is the OSQP parametrisation.

1.8 Equilibrium & CAPM (reprendre)

At equilibrium the sum of demand for stocks equals the sum of offered stocks which is equal to the market cap weighted portfolio which we denote α_m. If the aggregated demand of agents can be modeled by a representative agent (references ?) then the market portfolio shall be in proportion of the representative agent portfolio: $\pi = \lambda\Sigma\alpha$. Let π_m be the expected premium of the market portfolio and σ_m^2 its variance. It is clear that $\lambda = \frac{\pi_m}{\sigma_m^2}$. If we inject this value in the preceding equation and write it line by line we obtain:

$$\pi = \frac{\pi_m}{\sigma_m^2} \Sigma \alpha_m$$

or equivalently, line by line,

$$\forall i : \pi_i = \beta i \pi_m$$

since by definition

$$\beta_i = < e_i, \alpha_m >_\Sigma / \sigma_m^2$$

Comments:

- Return only related to the "common part" with the market portfolio
- no correlation => no return
- opposite of financial analysis

2 Dynamic hedging of options

2.1 Foundations

Typically a financial model is the combination of:

- a model of the spot price S_t,
- the bank account $B_t = e \int_0^t r_s ds$,
- the discount factor $D(t,T) = \frac{B(t)}{B(T)}$,
- the zero-coupon bond $P(t,T) = E(D(t,T) \mid \mathcal{F}_t)$
- a pricing measure Q under which ALL asset prices X_t obey the following equation

$$\frac{X_t}{B_t} = E\left(\frac{X_T}{B_T}|\mathcal{F}_t\right) \tag{7}$$

Theorem 4. *Future prices are martingale under the RNP while forward prices are martingale under the T-forward measure.*

From now on we will think in terms of future prices which are hence martingale under the RNP

$$S_t = E(S_T \mid \mathcal{F}_t)$$

and hence typically modelled as

$$dS_t = S_t \sigma(t) dW_t$$

where a non necessarily constant volatility might account for the part of interest rate risk in the future price dynamics.

2.2 Dynamic hedging & the "Greeks"

2.2.1 Representation theorem

Theorem 5. *Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$ a filtered space with Ft the natural filtration of some brownian motion W_t. For every F in $L^2(\mathcal{F}, \mathbb{Q})$ there exists a unique adapted process H_t such that*

$$E\left(\int_0^T H_s^2 ds\right) < +\infty$$

$$\begin{aligned} F &= E(F) + \int_0^T H_s dW_s \\ &\equiv E(F) + (H.W)_T \end{aligned}$$

Proof. See for instance Baudoin. Uniqueness follows from Ito isometry

Existence. Let K in $L^2(\mathcal{F}, \mathbb{Q})$ such that $\forall H$ acceptable, $E(K(H.W)_T) = 0$. This would be true in particular for all exponential of piecewise constant H and hence

$$\begin{aligned}\forall (t_0, .., t_n), \forall (\lambda_0, .., \lambda_n) : E\left(Ke^{\sum \lambda_i W_i}\right) = 0 \\ \iff & E(K|\ W_1, ..., W_n) = 0 \\ \Rightarrow & E(K \mid F) = 0 \\ \Rightarrow & K = 0\ \ \mathbb{Q}a.s.\end{aligned}$$

□

The consequence of the representation theorem is that any claim can be "represented" by a stochastic integral. Considering the future model, any claim can be represented or "hedged" by a dynamic strategy consisting in holding an appropriate number of stocks/future contracts.

European options are claims paid at date final date T So they are F measurable claims:

- $\Phi(S_T, k)$ where k is a vector of parameters and S an underlying
- asian options: $\Phi\left(\frac{1}{n}\sum_{i=1}^{n} S_{t_i}\right)$

$$\begin{aligned}X_t &= E(\Phi(S_T|\mathcal{F}_t)) \\ &= \varphi(t, S_t, y_t)\end{aligned}$$

where y_t might be a process of state variables. Here φ is itself the future price of an option written on a futures contract.

One can:

1. (Ito) differentiate w.r.t. state variables
2. Compute sensitivity to parameters

The less sophisticated a model, the less state variables.

2.2.2 Delta

Representation theorem

$$\exists \delta\ \mid X_t - X_0 = \int \delta_t dS_t$$

Locally, by Ito isometry

$$\delta_t = \frac{d\langle X, S\rangle}{d\langle S\rangle_t}$$

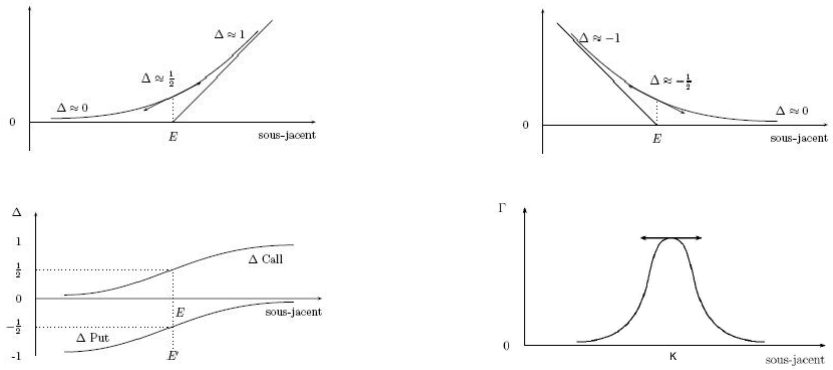


Figure 1: Some greeks in B&S model

is the regressor of X w.r.t. S. If we write

$$dX_t = \frac{\partial \varphi}{\partial s} dS_t + \frac{\partial \varphi}{\partial y} dy_t + \frac{1}{2} \frac{\partial^2 \varphi}{\partial s^2} d\langle S_t \rangle + ...dt$$
$$\delta = \frac{\partial \varphi}{\partial s}$$

Another way to approach it is to view it trajectory by trajectory:

$$\begin{aligned} \delta(t, S_t, y_t) =& \frac{\partial \varphi(t, S_t, y_t)}{\partial S_t} \\ =& \frac{\partial E(\Phi(S_T \mid \mathcal{F}_t))}{\partial S_t} \\ =& E\left(\frac{\partial \Phi}{\partial S_T} \frac{\partial S_T}{\partial S_t} \mid \mathcal{F}_t \right) \end{aligned}$$

In Black & Scholes

$$\delta = N(d_1)$$

Comments / Black-Scholes / Graphical / call option. Delta locally hedges the risk

2.2.3 Theta and Gamma

$$\theta \equiv \frac{\partial \varphi}{\partial t}$$
$$\gamma \equiv \frac{\partial^2 \varphi}{\partial x^2}$$

In case there is no other source of risk then

$$X_T - X_0 = \int_0^T \delta(t, S_t) \times dS_t + \frac{1}{2} \int_0^T \left(\theta_t + \frac{1}{2} \gamma \sigma_t^2 S_t^2 \right) dt$$

Since prices are martingale:

$$\theta_t + \frac{1}{2}\gamma\sigma_t^2 x^2 = 0$$
$$\varphi(T,x) = \Phi(x)$$

Hence the martingale property can be related to the permanent equality between gamma costs and theta depreciation. This is the financial equivalent of representation of conditional expectations as PDE solutions.

At any timeIn black & Scholes

$$\gamma = \frac{f(d_1)}{S_t\sigma\sqrt{T-t}}$$
$$\theta = -\frac{1}{2}\frac{S_t\sigma f(d_1)}{\sqrt{T-t}}$$

Price is only related to gamma !! Gamma costs

2.2.4 Vega

In B&S, volatility is a parameter so in that case one can compute:

$$vega = \frac{\partial\varphi(t,S_t,\sigma)}{\partial\sigma}$$
$$= E\left(\frac{\partial\Phi}{\partial\sigma} \mid \mathcal{F}_t\right)$$

Different from Ito differentiations, parameter sensitivity is the expectation of the derivative under the RNP.

In Black & Scholes

$$vega = S_t\sqrt{T-t}f(d_1)$$

2.3 Robustness of B&S formula for a European option

See nice article [6].

What if we are wrong in estimating the model volatility process ? Let $\sigma_E(t)$ the expected volatility process and $\sigma_R(t)$ the realized one. What is the result of dynamically hedging an option with this mistake? Let φ_E the estimated price function i.e. the price of the option with the improper model. φ_E is solution of the following PDE:

$$\frac{\partial\varphi_E}{\partial t} + \frac{1}{2}\frac{\partial\varphi_E^2}{\partial x^2}\sigma_E^2 x^2 = 0$$
$$\varphi_E(T,x) = \Phi(x)$$

The hedging strategy value yields

$$V(T) = \varphi_E(0,S_0) + \int_0^T \frac{\partial\varphi_E}{\partial x}dS_t$$

with

$$dS_t = S_t \sigma_R(t)\, dW_t$$

Now, Ito rules applied to φ_E shows that

$$\varphi_E(T, S_T) - \varphi_E(0, S_0) = \int_0^T \frac{\partial \varphi_E}{\partial x} dS_t + \int_0^T \left(\frac{\partial \varphi_E}{\partial t} + \frac{1}{2} \frac{\partial \varphi_E^2}{\partial x^2} \sigma_R^2 S_t^2 \right) dt$$

Now the difference between the final value of the hedging strategy $V(T)$ and the option payoff is obtained by substracting the two previous equations:

$$\begin{aligned} V(T) - \Phi(S_T) &= -\int_0^T \left(\frac{\partial \varphi_E}{\partial t} + \frac{1}{2} \frac{\partial \varphi_E^2}{\partial x^2} \sigma_R^2 S_t^2 \right) dt \\ &= \frac{1}{2} \int_0^T \frac{\partial \varphi_E^2}{\partial x^2} \left(\sigma_E^2 - \sigma_R^2 \right) S_t^2 dt \end{aligned}$$

Comments... Notions of being gamma plus, gamma neg etc.

3 Study of certain dynamic strategies

See [2] for a complete presentation.

In this section, we no longer adress the question from a normative perspective but try to make sense out of certain popular dynamic strategies. Our aim is to be able to answer questions such as:

- How is it possible for funds that have performed consistently to tumble in just a few months?
- Are these brutal reversals only attributable to market factors or do certain investment behaviors generate Extreme Risks?
- Which strategies benefit from market volatility?
- What are the best and worse possible scenarios for a given strategy?
- What risks are associated with mean-reversion strategies ?

3.1 General framework

We consider a simplified situation where interest rates are null and holdings only depend on the spot price plus some parameters.

The number of risky asset in portfolio depends on its spot price:

$$dV_t = f(S_t)\, dS_t$$

Naïve interpretation as a curve integral: the wealth is just the primitive of the holding function:

$$V_t - V_0 = \int_{S_0}^{S_t} f(x)\, dx$$

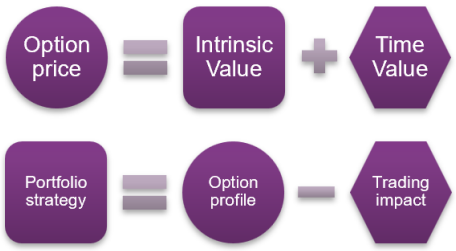


Figure 2: Parallel between option and portfolio strategies

Let F this function:

$$V_t - V_0 = F(S_T) - F(S_0)$$

This would be true if prices were deterministic. In the application of Ito to F yields

$$dF = f(S)\,dS + \frac{1}{2} f'(S)\, S^2 \sigma^2 dt$$

Hence the following result:

Theorem 6 (Decomposition theorem)**.** *Any portfolio strategy of the type (1) described above can be broken down into an option profile plus some trading impact as follows:*

$$V_t - V_0 = \underbrace{F(S_t) - F(S_0)}_{Option\ profile} - \underbrace{\frac{1}{2}\int_0^t f'(S_s)\, S_s^2 \sigma_s^2 ds}_{Trading\ Impact}$$

- No need for probabilities... (except no jump assumption)
- Path dependent trading impact, but European option profile.
- May depend of realized variance AND spot trajectory.

Let us now elaborate on these two terms. Trading impact depends on the variation in the number of holdings in relation to market fluctuations. If the strategy involves buying when the market goes up ($f' > 0$) then the trading impact will necessarily be negative, illustrating the "buy high sell low" curse of trend followers. Conversely, strategies that play against the market will always have positive trading impact, benefiting from the opposite effect. This trading impact increases with the volatility.

Interestingly, the sign of trading impact is directly related to the convexity of the option profile. Positive trading impact is necessarily associated with concave profile. This is no surprise to those familiar with option hedging, where it is well known that hedging a convex profile will generate positive gamma gains, which explains the difference between option prices and their intrinsic value. All

strategies with positive trading impact share similar characteristics in terms of winning probability. When trading impact is positive, using Theorem leads to

$$Pr\left(X_T > X_0\right) > Pr\left(F\left(S_T\right) > F\left(S_0\right)\right)$$

In a situation where F is non-decreasing, we get:

$$Pr\left(X_T > X_0\right) > Pr\left(S_T > S_0\right)$$

The probability of showing a profit is therefore higher than the probability of the underlying asset going up. On a weekly basis, most financial assets have a near 50/50 probability of going up or down, which means that those strategies have more chance of showing a profit than a loss. The higher the trading impact or the more concave the option profile, the higher the probability of showing a profit. This effect is offset by higher potential losses. A concave profile will therefore always exhibit negative skewness. Using a plain realised Sharpe ratio to assess future fund performance is very likely to be flawed, as it would be inflated artificially by the frequent positive gains. Following this analysis, rather than indicating a good portfolio manager, frequent positive gains may be symptomatic of strategies with high possible losses. The reverse holds for strategies with negative trading impact.This general framework allows to draw some general behavior from trading strategies as displayed in figure 3.

3.2 Contrarian strategies

3.2.1 Volatility statistical arbitrage

Imagine that you have a view that variance should come back to some kind of long-term value

$$\begin{aligned} d\sigma_t^2 &= \lambda\left(\sigma_\infty^2 - \sigma_t^2\right)dt + \sigma_t^2 \xi dW_t \\ \sigma_0^2 &= v_0 \end{aligned}$$

then a natural strategy would be to hold

$$\delta = k\left(\frac{\sigma_\infty^2}{\sigma_t^2} - 1\right)$$

where k encapsulates parameters λ and ξ. The strategy vega would be

$$vega = 2k\left(\frac{\sigma_\infty^2}{\sigma_t} - \sigma_t\right)$$

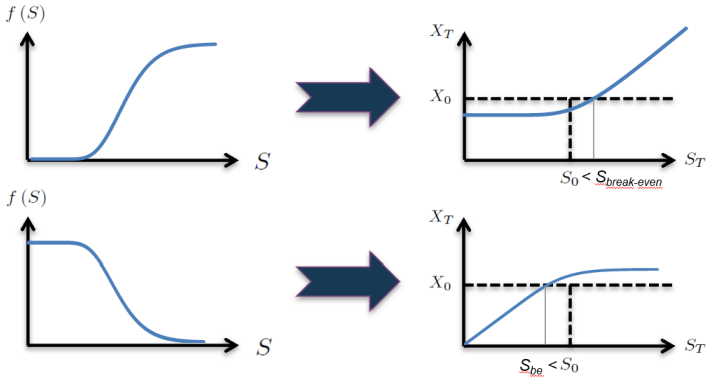
The option profile would be

$$F\left(\sigma^2\right) = k\left(\sigma_\infty^2 ln\left(\sigma^2\right) - \sigma^2\right)$$

and the strategy trading impact

$$\frac{1}{2}k\sigma_\infty^2\xi^2 T$$

CONVEX VS CONCAVE STRATEGIES



Strategy Type	Option Profile	Trading Impact	Hit Ratio	Average Gain / Loss Comparison
Buying when market goes up	Convex	Negative	$\lesssim 50\%$	Average Gain > Average Loss
Buying when market goes down	Concave	Positive	$\gtrsim 50\%$	Average Loss > Average Gain

Figure 3: Convex vs concave strategies

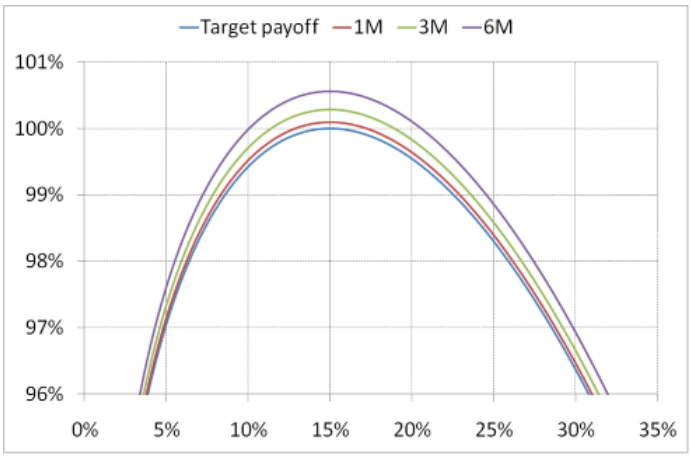


Figure 4: Volatility arbitrage profile

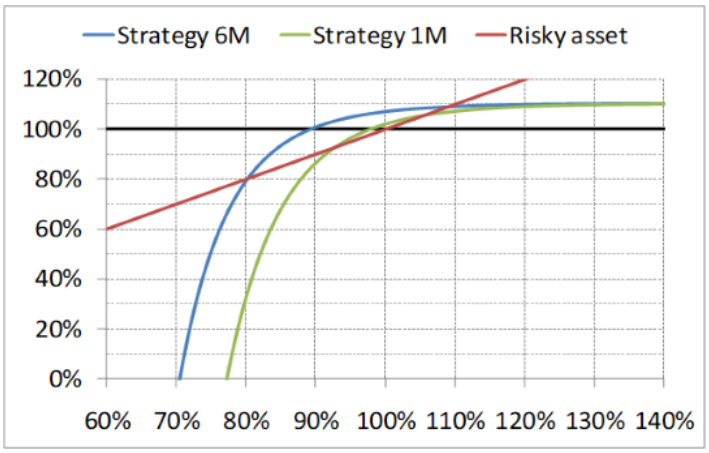


Figure 5: Doubling profile

3.2.2 Doubling strategy: "when in trouble, double"

The martingale strategy is to double the bet after every loss, so that the first win would recover all previous losses, plus a profit equal to the original stake. The gambler will almost surely win if he is allowed to bet an infinite number of times. Unfortunately, he may go bust before his first win in real life situations, as his stakes double at each loss. Indeed, after 5 consecutive losses, the gambler has to bet 32 times his original stake, which is unacceptable in most real life situations. The finite variance condition to qualify acceptable strategies also rules out this kind of strategies.

Having said that, let us study a "doubling strategy". Consider an exposure function α of wealth V which satisfies

$$\begin{aligned} &\forall V < V_{target} : V_{target} = V\left(1 + R_{target}\alpha\left(V\right)\right) \\ &e\left(V\right) < expo_{max} \\ &\forall V \geq V_{target} : \alpha\left(V\right) = 0 \end{aligned}$$

By construction, whenever markets move exceeds R_{target} the wealth exceeds V_{target}. .... This strategy does not belong to the type studied before but still can be integrated as

$$V_T = V_{target} - (V_{target} - V_0)\left(\frac{S_T}{S_0}\right)^{-1/R_{target}} exp\left(-\frac{1}{2}\left(\frac{1}{R_{target}} + \frac{1}{R_{target}^2}\right)\int_0^T \sigma_s^2 ds\right)$$

3.3 "Constant Mix" strategy

Might seem the simplest possible strategy

$$\frac{dV_t}{V_t} = e\frac{dS_t}{S_t}$$

Neither explicitely a delta hedging strategy since δ depends on the level of wealth. However it can be integrated as

$$V_t = V_0 \left(\frac{S_T}{S_0}\right)^e exp\left(\frac{1}{2}\int_0^T \left(e - e^2\right)\sigma_t^2 dt\right)$$

Analysis in terms of convexity, concavity, gamma costs etc.

Compare witht the martingale approach with log utility and B&S model.

3.4 Stop Loss vs Put option

See [3] for a complete treatment

Imagine a situation where an investor runs an investment strategy which value is denoted St. In order to limit its losses, this investor wants to add a separate overlay to the strategy, which would consist in doing nothing while the strategy is above a certain threshold level Sstop, and going short when it is below. In the following example, we will consider the initial strategy as an underlying asset, and focus on the stop-loss overlay analysis. This overlay is purely price dependent and should satisfy the assumptions of proposition 1. The corresponding function f, representing the number of risky asset shares in the overlay portfolio, would be:

$$f\left(S_t\right) = \begin{cases} 0 & \text{when } S_t > K \\ -1 & \text{when } S_t \leq K \end{cases}$$

The option profile would be a put option

$$F\left(S_t\right) = \left(S_t - K\right)^+$$

What would be the difference between the pay-off of the put option and the final value of the strategy. One can apply Ito formally to F and get

$$F\left(S_t\right) - F\left(S_0\right) = \int_0^t 1_{S_u > K} dS_u + \underbrace{\frac{1}{2}\int_0^t \sigma_t^2 S_t^2 \delta_K\left(S_t\right) dt}_{L_S(K,t)}$$

where δ_K is the Dirac operator in K. This second term is an accumulation of time spend at strike K by the spot amplified by the bracket of the price process. This term is strictly positive. Under the risk neutral probability

$$E\left(F\left(S_t\right)\right) = F\left(E\left(S_t\right)\right) + E\left(L_S\left(K,t\right)\right)$$

The time value of the option is the average local time at the strike. This strategy replication entails a risk which could have almost reduced to zero with the appropriate delta strategy.

4 Normative dynamic programming

This approach is usually referred to as the Samuelson-Merton due to the pioneering works of Samuelson ([9]) and Merton ([7]). We first remind the 1 period results and then move on to the dynamic state.

4.1 Dynamic Framework

4.1.1 HJB equation

Risky assets dynamics are described by the following SDE:

$$\begin{aligned} dS_t &= diag\left(S_t\right)\left(\pi\left(t,S_t\right)dt + \sigma\left(t,S_t\right)dW_t\right) \\ S\left(0\right) &= s \end{aligned}$$

with

$$\begin{cases} S_t, s, W_t, \mu_t \in R^{n\times 1} \\ \sigma_t \in R^{n\times n} \end{cases}$$

The wealth process is controlled by the percentage of wealth α_t^i invested in asset i. The local covariance matrix can be obtained as

$$\begin{aligned} \Sigma dt &\equiv E\left(dSdS'\right) \\ &= E\left(\sigma dW dW'\sigma'\right) \\ &= \sigma\sigma' dt \end{aligned}$$

Let $\alpha_t \in R^{n\times 1}$ the corresponding process. As in dimension 1, $\forall t : \alpha_t^0 = 1 - \sum \alpha_t^i$ is borrowed or invested into the bank account. We treat the case where $r = 0$ which means that π is a vector of premia and S are futures contracts. The wealth process can be written

$$\begin{aligned} dV_t &= V_t \sum \alpha_i \frac{dS_t^i}{S_t^i} \\ &= V_t\left(\alpha_t'\pi_t dt + \sum_i \alpha_i\left(\sum_j \sigma_{ij} dW_j\right)\right) \\ &= V_t\left(\alpha_t'\left(\pi_t dt + \sigma dW_t\right)\right) \\ V\left(t\right) &= v \end{aligned} \tag{8}$$

From the self financing equation 8 we get:

$$\begin{aligned} E\left(dV\right) &= V\alpha'\pi dt \\ E\left(dV^2\right) &= E\left(\alpha'\sigma dW_t dW_t'\sigma'\alpha\right) \\ &= \alpha'\Sigma\alpha dt \end{aligned}$$

The set of all admissible portfolio strategies is defined as

$$\mathcal{A}_{[t_1,t_2]} = \left\{\forall t_1 \le t \le t_2 : \alpha_t \in A \; p.s. \; , A \text{ closed}\right\}$$

We seek to solve the following problem:

$$u(t,s,v) = \sup_{\alpha \in \mathcal{A}_{[t,T]}} E\left(U\left(V_T(\alpha)\right) \mid S_t = s, V_t = v\right) \tag{9}$$

The resolution upon the following theorem:

Theorem 7 (Principle of dynamic programming)**.** *If*

$$u(t,s,v) \equiv \sup_{\alpha_t \in \mathcal{A}_{[t,T]}} E\left(U\left(V_T\right) \mid S_t = s, V_t = v\right)$$

then

$$\forall h < T : u(t,s,v) = \sup_{\alpha_t \in \mathcal{A}_{[t,t+h]}} E\left(u\left(s_h, v_h, t+h\right) \mid S_t = s, V_t = v\right)$$

Loosely speaking, this principle states that any subpath of an optimal path is optimal. It is sometimes called the "maximum" principle since, in the deterministic case, the proof consists in "cutting" the control $\alpha_{0\leq t\leq T}$ into two sub-controls $\alpha_1 \equiv \alpha_{0\leq t\leq t_1}$ and $\alpha_2 = \alpha_{t_1\leq t\leq T}$ and acknowledging that

$$\sup_{\alpha} F(\alpha) = \sup_{\alpha_1,\alpha_2} F(\alpha_1,\alpha_2) = \sup_{\alpha_1}\left(\sup_{\alpha_2} F(\alpha_1,\alpha_2)\right)$$

The Hamilton Jacobi Bellman equation is obtained by combining Ito and the principle of dynamic programming. Let us consider the case were the value fonction does not depend on S_t

$$u(t+h, V_{t+h}) = u(t,V_t) + h\frac{\partial u}{\partial t}(t,V_t) + (V_{t+h}-V_t)\frac{\partial u}{\partial v}(t,V_t) + \frac{1}{2}(V_{t+h}-V_t)^2\frac{\partial^2 u}{\partial v^2}(t,V_t) + o\left(\|V_{t+h}-V_t\|^2\right)$$

Now, the principle of dynamic programming ensures that

$$u(t,V_t) = \sup_{\alpha \in \mathcal{A}_{[t,t+h]}} E\left(u\left(s_h, v_h, t+h\right) \mid S_t = s, V_t = v\right)$$

$$\Rightarrow u(t,V_t) = \sup_{\alpha \in \mathcal{A}_{[t,t+h]}} E\left(u(t,V_t) + h\frac{\partial u}{\partial t}(t,V_t) + (V_{t+h}-V_t)\frac{\partial u}{\partial v}(t,V_t) + \frac{1}{2}(V_{t+h}-V_t)^2\frac{\partial^2 u}{\partial v^2}(t,V_t) + o\left(\|V_{t+h}\right.\right.$$

In the one dimensional case:

$$E\left((V_{t+h}-V_t)\,|V_t,\alpha_t\right) \approx V_t\alpha_t\pi_t h + o(h)$$
$$E\left((V_{t+h}-V_t)^2\,|V_t,\alpha_t\right) \approx V_t^2\alpha_t^2\sigma_t^2 h + o(h)$$

and hence

$$0 = h \times \sup_{\alpha \in \mathcal{A}_{[t,t+h]}}\left(\frac{\partial u}{\partial t} + V_t\alpha_t\pi_t\frac{\partial u}{\partial v} + \frac{1}{2}V_t^2\alpha_t^2\sigma_t^2\frac{\partial^2 u}{\partial v^2}\right) + o(h)$$

Which at the limits yields

$$\begin{cases} \frac{\partial u}{\partial t} + \sup\limits_{\alpha \in R} \left(V_t \alpha \pi_t \frac{\partial u}{\partial v} + \frac{1}{2} V_t^2 \alpha^2 \sigma_t^2 \frac{\partial^2 u}{\partial v^2} \right) = 0 \\ u(T, V_T) = U(V_T) \end{cases} \tag{10}$$

Similarly, in the n-dimensional case

$$\begin{cases} \frac{\partial u}{\partial t} + \sup\limits_{\alpha \in R^n} \left(V \alpha' \pi_t \frac{\partial u}{\partial v} + \frac{1}{2} V_t^2 \frac{\partial^2 u}{\partial v^2} \alpha' \Sigma_t \alpha \right) = 0 \\ u(T, V_T) = U(V_T) \end{cases} \tag{11}$$

This is summarized in the following verification theorem

Theorem 8. *If $u(t,x)$ is $C^{1,1}$ and satisfies equation 11 then it is the solution of program 9.*

Now, let $\lambda(t, V_t) \equiv -V \frac{\partial^2 u}{\partial v^2} / \frac{\partial u}{\partial v}$ then the optimal strategies is attained at:

$$\alpha_t = \frac{1}{\lambda(t, V_t)} \Sigma_t^{-1} \pi_t$$

Comments. Myopic behavior. Series of Markowitz.

4.1.2 CARA solution

Let $U(x) = -e^{-\lambda x}$. Let us look at the solution as

$$\begin{aligned} u(t, v) &= U(v + f(t)) \\ &= -exp(-\lambda(v + f(t))) \end{aligned}$$

In that case

$$\begin{aligned} \frac{\partial u}{\partial t} &= -\lambda f' u \\ \frac{\partial u}{\partial v} &= -\lambda u \\ \frac{\partial^2 u}{\partial v^2} &= \lambda^2 u \end{aligned}$$

Injecting in 11 and being mindful of the (negative) sign of u:

$$\lambda f' + \sup_{\alpha \in R^n} \left(\lambda V \alpha' \pi_t - \frac{1}{2} V_t^2 \lambda^2 \alpha' \Sigma_t \alpha \right) = 0$$

One gets

$$\alpha = \frac{1}{\lambda V} \Sigma^{-1} \pi$$

Comments. "ratchet effect"

Then

$$\lambda f' + \frac{\pi' \Sigma^{-1} \pi}{2} = 0$$
$$f(T) = 0$$

and hence

$$f(t) = \frac{\pi' \Sigma^{-1} \pi}{2\lambda} (T - t)$$

Comments. Utility benefit independent of wealth. $\pi' \Sigma^{-1} \pi$ as a norm....

4.1.3 CRRA solution

Let $U(x) = x^{1-\gamma}$. Let us look at the solution as

$$u(t, v) = U(v + f(t, T))$$
$$= v^{1-\gamma} f^{1-\gamma}(t)$$

In that case

$$\frac{\partial u}{\partial t} = (1 - \gamma) \frac{f'}{f} u$$
$$\frac{\partial u}{\partial v} = (1 - \gamma) \frac{u}{v}$$
$$\frac{\partial^2 u}{\partial v^2} = (1 - \gamma) \gamma \frac{u}{v^2}$$

Injecting in 11:

$$\frac{f'}{f} + \sup_{\alpha \in R^n} \left(\alpha' \pi_t - \frac{1}{2} \gamma \alpha' \Sigma_t \alpha \right) = 0$$

One gets

$$\alpha = \frac{1}{\gamma} \Sigma^{-1} \pi$$

Then

$$\frac{f'}{f} + \frac{\pi' \Sigma^{-1} \pi}{2\gamma} = 0$$
$$f(T) = 0$$

and hence

$$f(t) = exp\left(\frac{\pi' \Sigma^{-1} \pi}{2\gamma} (T - t) \right)$$

Comments. Utility benefit proportional of wealth. $\pi' \Sigma^{-1} \pi$ as a return...

5 Martingale approach, the bridge between options and portfolio management

5.1 Introduction

We consider a model with possibly several assets but, for the sake of simplicity, we expose it with only one asset.

$$\begin{aligned} dS_t &= S_t \left(\mu \left(t, S_t \right) dt + \sigma \left(t, S_t \right) dW_t^P \right) \\ S \left(0 \right) &= s \end{aligned}$$

The above equation is written under the subjective probability of the agent. μ and σ contain his expectations about future returns. The agent wants to run a dynamic strategy which consists in holding a fraction α_t of its wealth V_t. The remaining wealth $\left(1 - \alpha_t \right) V_t$ is put or borrowed into the bank account. The wealth dynamics yields

$$\begin{aligned} dV_t &= V_t \left(\alpha_t \frac{dS_t}{S_t} + \left(1 - \alpha_t \right) \frac{dB_t}{B_t} \right) \\ &= V_t \left(\alpha_t \pi_t dt + \alpha_t \sigma_t dW_t \right) \\ V \left(0 \right) &= v \end{aligned}$$

Another way to parametrize the strategies is to think in terms of delta i.e. number of stocks instead of proportion of wealth:

$$\delta_t dS_t = \alpha_t V_t$$

In that case the wealth equation becomes

$$\begin{aligned} dV_t &= \delta_t dS_t \\ V \left(0 \right) &= v \end{aligned}$$

Now, whatever the description of the dynamic strategy, the agent is supposed to maximise the average value of its utility function at a future date T:

$$u \left(v, s \right) = \sup_{\substack{\alpha \text{ acceptable} \\ \delta \text{ acceptable}}} E \left(U \left(V_T \right) \right)$$

where the concept of acceptable strategy has to be precised. There are two different routes to solve this problem.

5.2 Theory

The martingale approach has been pioneered by Cox-Huang [4], Karatzas et al. [5]. See also [10].

Instead of controlling the strategy, the idea is to control the final result. By No arbitrage, it is equivalent to choose an option payoff or its delta hedging strategy. If the set of acceptable delta strategy is the set of square integrable adapted processes, then the representation theorm allow to state that:

$$u(v) = \begin{cases} \underset{\mathrm{X}\in L^2(\Omega,\mathcal{F},P)}{sup} & E^P(U(X)) \\ E^{Q_T}(X) = v/P(0,T) \end{cases}$$

There exists a Lagrange constant k such that

$$u(v) = \underset{\mathrm{X}\in L^2(\Omega,\mathcal{F},P)}{sup} \mathbb{E}^{\mathbb{P}}(U(X)) - k\left(\mathbb{E}^{\mathbb{Q}_T}(X) - v^f\right)$$

where $v^f = v/P(0,T)$. Let $H_T = \frac{dQ_T}{dP}$ the above can be rewritten

$$u(v) = \underset{\mathrm{X}\in L^2(\Omega,\mathcal{F},P)}{sup} \mathbb{E}^{\mathbb{P}}(U(X) - kXH_T) - kv^f$$

Let solve this program heuristically. Let $\varphi_Y(\epsilon) = \mathbb{E}^{\mathbb{P}}(U(X+\epsilon Y) - k(X+\epsilon Y)H_T)$.

$$X^* \text{ optimal} \Longleftrightarrow \forall Y : \frac{d\varphi_Y(\epsilon)}{d\epsilon}|_{\epsilon=0} = 0$$

Now $\frac{d\varphi_Y(\epsilon)}{d\epsilon} = \mathbb{E}^{\mathbb{P}}(YU'(X+\epsilon Y) - kYH_T)$ and $\frac{d\varphi_Y(\epsilon)}{d\epsilon}|_{\epsilon=0} = \mathbb{E}^{\mathbb{P}}(Y(U'(X) - kH_T))$. For this quantity to be equal to zero $\forall Y$, one necessearily has

$$U'(X) = kH_T$$

In case U' can be inversed we denote $I(x) \equiv U'^{-1}(x)$. Then

$$X^* = I(kH_T)$$

where k is obtained thanks to the budget constraint.

5.3 B&S expectations and logarithmic utility

For a logarithmic utility $U(x) = ln(x)$, $I(x) = \frac{1}{x}$ and hence

$$X^* = k\frac{d\mathbb{P}}{d\mathbb{Q}_T}$$

which, using the budget constraint yields

$$X^* = v\frac{d\mathbb{P}}{d\mathbb{Q}_T}$$

The value of this information is given by the gain in terms of utility

$$E^{\mathbb{P}}\left(ln\left(\frac{d\mathbb{P}}{d\mathbb{Q}_T}\right)\right) - ln\left(E^{\mathbb{Q}}\left(\frac{d\mathbb{P}}{d\mathbb{Q}_T}\right)\right) = E^{\mathbb{Q}}\left(\frac{d\mathbb{P}}{d\mathbb{Q}_T}ln\left(\frac{d\mathbb{P}}{d\mathbb{Q}_T}\right)\right)$$

which is the divergence or entropy or Kullback-Leibler distance between the expectation and the Risk Neutral Probability.The probability density can be seen as a portfolio ! Let us study some specific cases.

In a B&S framework

$$\begin{aligned} dS_t &= S_t \sigma dW_t^{\mathbb{Q}} \\ &= S_t \sigma \left(dW_t^{\mathbb{P}} + \lambda dt \right) \end{aligned}$$

where λ is the expected Sharpe ratio of the asset. Let $H_t = \mathbb{E}^{\mathbb{Q}} \left(X^* \mid \mathcal{F}_t \right)$. We have

$$dH_t = \lambda H_t dW_t^{\mathbb{Q}}$$

and we can check that

$$\begin{aligned} dW_t^{\mathbb{Q}} - d \left\langle ln\left(H\right), W^{\mathbb{Q}} \right\rangle_t &= dW_t^{\mathbb{Q}} - \lambda dt \\ &= dW_t^{\mathbb{P}} \end{aligned}$$

Hence

$$\begin{aligned} H_t &= exp\left(\lambda W_t^{\mathbb{Q}} - \frac{1}{2} \lambda^2 t \right) \\ S_t &= exp\left(\sigma W_t^{\mathbb{Q}} - \frac{1}{2} \sigma^2 t \right) \\ H_t &= S_t^{\frac{\lambda}{\sigma}} \times exp\left(-\frac{1}{2} \sigma^2 \left(\left(\frac{\lambda}{\sigma} \right)^2 - \frac{\lambda}{\sigma} \right) t \right) \end{aligned} \tag{12}$$

So the optimal strategy is to hedge a power option on the underlying. The corresponding delta hedging strategy is equal to

$$\begin{aligned} \delta_t = \frac{d\left\langle H, S \right\rangle}{\left\langle S \right\rangle} &= \frac{\lambda \sigma H S}{\sigma^2 S^2} \\ &= \frac{\lambda}{\sigma} \frac{H_t}{S_t} \end{aligned}$$

In such case the amount invested as a fraction of the wealth H_t is equal to

$$\alpha_t = \frac{\delta_t S_t}{H_t} = \frac{\lambda}{\sigma} = cte$$

and hence

$$\frac{dH}{H} = \frac{\lambda}{\sigma} \frac{dS}{S}$$

The terms in 12 can be understood as "gamma" terms. One indeed gets:

$$\begin{aligned} d\delta &= \frac{\lambda}{\sigma} \left(\frac{dH_t}{S_t dS} - \frac{H}{S^2} dS \right) \\ &= \frac{H}{S^2} \left(\left(\frac{\lambda}{\sigma} \right)^2 - \frac{\lambda}{\sigma} \right) dS \end{aligned}$$

from which

$$d\langle \delta S \rangle = H\left(\left(\frac{\lambda}{\sigma}\right)^2 - \frac{\lambda}{\sigma}\right)\sigma^2 dt$$

Term similar to the robustness term. Implicit "gamma" of the constant mix strategy.

5.4 B&S expectations and general utility

U utility, I increasing function.

Increasing function of a power pay-off

5.5 Logarithmic utility and "European" expectation

5.5.1 General framework

See [1] for a complete treatment

Imagine now that the agent has an expectation regarding the value of an underlying at a certain date T. Typically a stock is likely to have a certain price objective of but with a negative scenario etc. This might be modelled as a distribution probability $\mathbb{P}$ which density w.r.t. $\mathbb{Q}$ is a function of the log return W_T.

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = \Phi(W_T)$$
$$E^{\mathbb{Q}}(\Phi(W_T)) = 1$$

At any time we denote

$$H_t = E^{\mathbb{Q}}(\Phi(W_T) \mid \mathcal{F}_t)$$

the conditional density of this expectation. The optimal delta strategy w.r.t. to the brownian motion consists in holding

$$\delta_t = E^{\mathbb{Q}}\left(\frac{\partial W_T}{\partial S_t}\Phi'(W_T) \mid \mathcal{F}_t\right)$$

In our model, under the RNP

$$\begin{aligned}\frac{\partial W_T}{\partial S_t} &= \frac{\partial W_T}{\partial W_t}\frac{\partial W_t}{\partial S_t} \\ &= \frac{1}{\sigma S_t}\end{aligned}$$

and hence

$$\begin{aligned}\delta_t S_t &= \frac{1}{\sigma}E^{\mathbb{Q}}(\Phi'(W_T) \mid \mathcal{F}_t) \\ &= \frac{1}{\sigma}E^{\mathbb{Q}}\left(\frac{\Phi(W_T)}{\Phi(W_T)}\Phi'(W_T) \mid \mathcal{F}_t\right) \\ &= \frac{1}{\sigma}E^{\mathbb{P}}\left(\frac{\Phi'(W_T)}{\Phi(W_T)} \mid \mathcal{F}_t\right) \times H_t\end{aligned} \tag{13}$$

which is equivalent to

$$\alpha_t = \frac{\delta_t S_t}{H_t} = E^{\mathbb{P}}\left(\frac{\Phi'(W_T)}{\Phi(W_T)} \mid \mathcal{F}_t\right) \tag{14}$$

Depending on circumstances, equation 13 or 14 can be used to compute the optimal portfolio strategy.

5.5.2 Gaussian expectations

Let us consider the case where the agent's expectation regarding the Sharpe ratio is gaussian, with mean $m\%$ p.a. and variance s^2. Then the density w.r.t. to the risk neutral probability is

$$\Phi(x) = \frac{\sqrt{T}}{s} e^{-\frac{1}{2}\left(\frac{(x-mT)^2}{s^2} - \frac{x^2}{T}\right)}$$

$$\Phi'(x) = \left(\frac{mT-x}{s^2} + \frac{x}{T}\right)\Phi(x)$$

Proposition 1. *Let $\gamma \equiv \frac{s^2}{T} - 1$ the "precision" of the information . The optimal portfolio strategy in the Gaussian case is obtained as*

$$\alpha_t = \frac{1}{\sigma}\frac{\gamma W_t + mT^2}{\gamma t + T}$$

Proof. Using equation 13 it comes that:

$$\begin{aligned}\delta_t S_t &= \frac{1}{\sigma} E^{\mathbb{Q}}\left(\Phi(W_T) \mid \mathcal{F}_t\right) \\ &= \frac{1}{\sigma}\frac{\sqrt{T}}{s}\int_{\mathbb{R}}\left(\frac{mT-x}{s^2} + \frac{x}{T}\right) e^{-\frac{1}{2}\left(\frac{(x-mT)^2}{s^2} - \frac{x^2}{T}\right)} \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{1}{2}\frac{(x-W_t)^2}{T-t}} dx\end{aligned}$$

The second order polynomial in the exponential must be written in its canonical form. lengthy calculations yield

$$\delta_t S_t = \frac{1}{\sigma}\frac{\left(s^2 - T\right)W_t + mT^2}{\left(s^2 - T\right)t + T^2} \times E^{\mathbb{Q}}\left(\Phi(W_T) \mid \mathcal{F}_t\right)$$

and hence the result

Comments: □

1. Case $\gamma = -1$, the brownian bridge case
2. Case $\gamma = 0$,the constant-mix
3. Remarks on fat tails

6 Parameters estimation

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